2

Fundamentals of Logic

In the first chapter we derived a summation formula in Example 1.40 (Section 1.4). We obtained this formula by counting the same collection of objects (the statements that were executed in a certain program segment) in two different ways and then equating the results. Consequently, we say that the formula was established by a combinatorial proof. This is one of many different techniques for arriving at a proof.

In this chapter we take a close look at what constitutes a valid argument and a more conventional proof. When a mathematician wishes to provide a proof for a given situation, he or she must use a system of logic. This is also true when a computer scientist develops the algorithms needed for a program or system of programs. The logic of mathematics is applied to decide whether one statement follows from, or is a logical consequence of, one or more other statements.

Some of the rules that govern this process are described in this chapter. We shall use these rules in proofs (provided in the text and required in the exercises) throughout subsequent chapters. However, at no time can we hope to arrive at a point at which we can apply the rules in an automatic fashion. As in applying the counting ideas discussed in Chapter 1, we should always analyze and seek to understand the situation given. This often calls for attributes we cannot learn in a book, such as insight and creativity. Merely trying to apply formulas or invoke rules will not get us very far either in proving results (such as theorems) or in doing enumeration problems.

2.1 Basic Connectives and Truth Tables

In the development of any mathematical theory, assertions are made in the form of sentences. Such verbal or written assertions, called *statements* (or *propositions*), are declarative sentences that are either true or false — but *not* both. For example, the following are statements, and we use the lowercase letters of the alphabet (such as p, q, and r) to represent these statements.

p: Combinatorics is a required course for sophomores.

q: Margaret Mitchell wrote *Gone with the Wind*.

r: $2 + 3 = 5$.

On the other hand, we do not regard sentences such as the exclamation

"What a beautiful evening!"

or the command

"Get up and do your exercises."

as statements since they do not have *truth values* (true or false).

The preceding statements represented by the letters p, q, and r are considered to be *primitive* statements, for there is really no way to break them down into anything simpler. New statements can be obtained from existing ones in two ways.

1) Transform a given statement p into the statement $\neg p$, which denotes its *negation* and is read "Not p."

For the statement p above, $\neg p$ is the statement "Combinatorics is not a required course for sophomores." (We do not consider the negation of a primitive statement to be a primitive statement.)

2) Combine two or more statements into a *compound* statement, using the following *logical connectives.*

a) Conjunction: The *conjunction* of the statements p, q is denoted by $p \wedge q$, which is read "p and q." In our example the compound statement $p \wedge q$ is read "Combinatorics is a required course for sophomores, **and** Margaret Mitchell wrote *Gone with the Wind*."

b) Disjunction: The expression $p \vee q$ denotes the *disjunction* of the statements p, q and is read "p or q." Hence "Combinatorics is a required course for sophomores, **or** Margaret Mitchell wrote *Gone with the Wind*" is the verbal translation for $p \vee q$, when p, q are as above. We use the word "or" in the *inclusive* sense here. Consequently, $p \vee q$ is true if one or the other of p, q is true or if *both* of the statements p, q are true. In English we sometimes write "and/or" to point this out. The *exclusive* "or" is denoted by $p \veebar q$. The compound statement $p \veebar q$ is true if one or the other of p, q is true but *not both* of the statements p, q are true. One way to express $p \veebar q$ for the example here is "Combinatorics is a required course for sophomores, or Margaret Mitchell wrote *Gone with the Wind*, but not both."

c) Implication: We say that "p implies q" and write $p \rightarrow q$ to designate the statement, which is the *implication* of q by p. Alternatively, we can also say

(i) "If p, then q." **(ii)** "p is *sufficient* for q."
(iii) "p is a *sufficient condition* for q." **(iv)** "q is *necessary* for p."
(v) "q is a *necessary condition* for p." **(vi)** "p only if q."

A verbal translation of $p \rightarrow q$ for our example is "If combinatorics is a required course for sophomores, then Margaret Mitchell wrote *Gone with the Wind*." The statement p is called the *hypothesis* of the implication; q is called the *conclusion*. When statements are combined in this manner, there need not be any causal relationship between the statements for the implication to be true.

d) Biconditional: Last, the *biconditional* of two statements p, q, is denoted by $p \leftrightarrow q$, which is read "p if and only if q," or "p is necessary and sufficient for q." For our p, q, "Combinatorics is a required course for sophomores if and only if Margaret Mitchell wrote *Gone with the Wind*" conveys the meaning of $p \leftrightarrow q$. We sometimes abbreviate "p if and only if q" as "p iff q."

Throughout our discussion on logic we must realize that a sentence such as

"The number x is an integer."

is *not* a statement because its truth value (true or false) cannot be determined until a numerical value is assigned for x. If x were assigned the value 7, the result would be a true statement. Assigning x a value such as $\frac{1}{2}$, $\sqrt{2}$, or π, however, would make the resulting statement false. (We shall encounter this type of situation again in Sections 2.4 and 2.5 of this chapter.)

In the foregoing discussion, we mentioned the circumstances under which the *compound* statements $p \vee q$, $p \veebar q$ are considered true, on the basis of the truth of their components p, q. This idea of the truth or falsity of a compound statement being dependent only on the truth values of its components is worth further investigation. Tables 2.1 and 2.2 summarize the truth and falsity of the negation and the different kinds of compound statements on the basis of the truth values of their components. In constructing such *truth tables*, we write "0" for false and "1" for true.

Table 2.1

p	$\neg p$
0	1
1	0

Table 2.2

p	q	$p \wedge q$	$p \vee q$	$p \veebar q$	$p \rightarrow q$	$p \leftrightarrow q$
0	0	0	0	0	1	1
0	1	0	1	1	1	0
1	0	0	1	1	0	0
1	1	1	1	0	1	1

The four possible truth assignments for p, q can be listed in any order. For later work, the particular order presented here will prove useful.

We see that the columns of truth values for p and $\neg p$ are the opposite of each other. The statement $p \wedge q$ is true only when both p, q are true, whereas $p \vee q$ is false only when both the component statements p, q are false. As we noted before, $p \veebar q$ is true when exactly one of p, q is true.

For the implication $p \rightarrow q$, the result is true in all cases except where p is true and q is false. We do not want a true statement to lead us into believing something that is false. However, we regard as true a statement such as "If $2 + 3 = 6$, then $2 + 4 = 7$," even though the statements "$2 + 3 = 6$" and "$2 + 4 = 7$" are both false.

Finally, the biconditional $p \leftrightarrow q$ is true when the statements p, q have the same truth value and is false otherwise.

Now that we have been introduced to certain concepts, let us investigate a little further some of these initial ideas about connectives. Our first two examples should prove useful for such an investigation.

EXAMPLE 2.1

Let s, t, and u denote the following primitive statements:

s: Phyllis goes out for a walk.

t: The moon is out.

u: It is snowing.

The following English sentences provide possible translations for the given (symbolic) compound statements.

a) $(t \wedge \neg u) \rightarrow s$: If the moon is out and it is not snowing, then Phyllis goes out for a walk.

b) $t \rightarrow (\neg u \rightarrow s)$: If the moon is out, then if it is not snowing Phyllis goes out for a walk. [So $\neg u \rightarrow s$ is understood to mean $(\neg u) \rightarrow s$ as opposed to $\neg(u \rightarrow s)$.]

c) $\neg(s \leftrightarrow (u \vee t))$: It is not the case that Phyllis goes out for a walk if and only if it is snowing or the moon is out.

Now we will work in reverse order and examine the logical (or symbolic) notation for three given English sentences:

d) "Phyllis will go out walking if and only if the moon is out." Here the words "if and only if" indicate that we are dealing with a biconditional. In symbolic form this becomes $s \leftrightarrow t$.

e) "If it is snowing and the moon is not out, then Phyllis will not go out for a walk." This compound statement is an implication where the hypothesis is also a compound statement. One may express this statement in symbolic form as $(u \wedge \neg t) \rightarrow \neg s$.

f) "It is snowing but Phyllis will still go out for a walk." Now we come across a new connective — namely, *but*. In our study of logic we shall follow the convention that the connectives *but* and *and* convey the same meaning. Consequently, this sentence may be represented as $u \wedge s$.

Now let us return to the results in Table 2.2, particularly the sixth column. For if this is one's first encounter with the truth table for the implication $p \rightarrow q$, then it may be somewhat difficult to accept the stated entries — especially the results in the first two rows (where p has the truth value 0). The following example should help make these truth value assignments easier to grasp.

EXAMPLE 2.2

Consider the following scenario. It is almost the week before Christmas and Penny will be attending several parties that week. Ever conscious of her weight, she plans not to weigh herself until the day after Christmas. Considering what those parties may do to her waistline by then, she makes the following resolution for the December 26 outcome: "If I weigh more than 120 pounds, then I shall enroll in an exercise class."

Here we let p and q denote the (primitive) statements

p: I weigh more than 120 pounds.

q: I shall enroll in an exercise class.

Then Penny's statement (implication) is given by $p \rightarrow q$.

We shall consider the truth values of this particular example of $p \rightarrow q$ for the rows of Table 2.2. Consider first the easier cases in rows 4 and 3.

- Row 4: p and q both have the truth value 1. On December 26 Penny finds that she weighs more than 120 pounds and promptly enrolls in an exercise class, just as she said she would. Here we consider $p \rightarrow q$ to be true and assign it the truth value 1.
- Row 3: p has the truth value 1, q has the truth value 0. Now that December 26 has arrived, Penny finds her weight to be over 120 pounds, but she makes no attempt to enroll in an exercise class. In this case we feel that Penny has broken her resolution — in other words, the implication $p \rightarrow q$ is false (and has the truth value 0).

The cases in rows 1 and 2 may not immediately agree with our intuition, but the example should make these results a little easier to accept.

- Row 1: p and q both have the truth value 0. Here Penny finds that on December 26 her weight is 120 pounds or less and she does not enroll in an exercise class. She has not violated her resolution; we take her statement $p \rightarrow q$ to be true and assign it the truth value 1.
- Row 2: p has the truth value 0, q has the truth value 1. This last case finds Penny weighing 120 pounds or less on December 26 but still enrolling in an exercise class. Perhaps her weight is 119 or 120 pounds and she feels this is still too high. Or maybe she wants to join an exercise class because she thinks it will be good for her health. No matter what the reason, she has not gone against her resolution $p \rightarrow q$. Once again, we accept this compound statement as true, assigning it the truth value 1.

Our next example discusses a related notion: the *decision* (or *selection*) structure in computer programming.

EXAMPLE 2.3

In computer science the **if-then** and **if-then-else** decision structures arise (in various formats) in high-level programming languages such as Java and C++. The hypothesis p is often a relational expression such as $x > 2$. This expression then becomes a (logical) statement that has the truth value 0 or 1, depending on the value of the variable x at that point in the program. The conclusion q is usually an "executable statement." (So q is not one of the logical statements that we have been discussing.) When dealing with "**if** p **then** q," in this context, the computer executes q only on the condition that p is true. For p false, the computer goes to the next instruction in the program sequence. For the decision structure "**if** p **then** q **else** r," q is executed when p is true and r is executed when p is false.

Before continuing, a word of caution: Be careful when using the symbols $\rightarrow$ and $\leftrightarrow$. The implication and the biconditional are not the same, as evidenced by the last two columns of Table 2.2.

In our everyday language, however, we often find situations where an implication is used when the intention actually calls for a biconditional. For example, consider the following implications that a certain parent might direct to his or her child.

$s \rightarrow t$: If you do your homework, then you will get to watch the baseball game.

$t \rightarrow s$: You will get to watch the baseball game only if you do your homework.

- Case 1: The implication $s \rightarrow t$. When the parent says to the child, "If you do your homework, then you will get to watch the baseball game," he or she is trying a positive approach by emphasizing the enjoyment in watching the baseball game.
- Case 2: The implication $t \rightarrow s$. Here we find the negative approach and the parent who warns the child in saying, "You will get to watch the baseball game only if you do your homework." This parent places the emphasis on the punishment (lack of enjoyment) to be incurred.

In either case, the parent probably wants his or her implication — be it $s \rightarrow t$ or $t \rightarrow s$ — to be understood as the biconditional $s \leftrightarrow t$. For in case 1 the parent wants to hint at the punishment while promising the enjoyment; in case 2, where the punishment has been used (perhaps, to threaten), if the child does in fact do the homework, then that child will definitely be given the opportunity to enjoy watching the baseball game.

In scientific writing one must make every effort to be unambiguous — when an implication is given, it ordinarily cannot, and should not, be interpreted as a biconditional. Definitions are a notable exception, which we shall discuss in Section 2.5.

Before we continue let us take a step back. When we summarized the material that gave us Tables 2.1 and 2.2, we may not have stressed enough that the results were for any statements p, q — *not* just primitive statements p, q. Examples 2.4 through 2.6 should help to reinforce this.

EXAMPLE 2.4

Let us examine the truth table for the compound statement "Margaret Mitchell wrote *Gone with the Wind*, and if $2 + 3 \neq 5$, then combinatorics is a required course for sophomores." In symbolic notation this statement is written as $q \wedge (\neg r \rightarrow p)$, where p, q, and r represent the primitive statements introduced at the start of this section. The last column of Table 2.3 contains the truth values for this result. We obtained these truth values by using the fact that the conjunction of any two statements is true if and only if both statements are true. This is what we said earlier in Table 2.2, and now one of our statements — namely, the implication $\neg r \rightarrow p$ — is definitely a compound statement, not a primitive one. Columns 4, 5, and 6 in this table show how we build the truth table up by considering smaller parts of the compound statement and by using the results from Tables 2.1 and 2.2.

Table 2.3

p	q	r	$\neg r$	$\neg r \rightarrow p$	$q \wedge (\neg r \rightarrow p)$
0	0	0	1	0	0
0	0	1	0	1	0
0	1	0	1	0	0
0	1	1	0	1	1
1	0	0	1	1	0
1	0	1	0	1	0
1	1	0	1	1	1
1	1	1	0	1	1

EXAMPLE 2.5

In Table 2.4 we develop the truth tables for the compound statements $p \vee (q \wedge r)$ (column 5) and $(p \vee q) \wedge r$ (column 7).

Table 2.4

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$(p \vee q) \wedge r$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	1	0
0	1	1	1	1	1	1
1	0	0	0	1	1	0
1	0	1	0	1	1	1
1	1	0	0	1	1	0
1	1	1	1	1	1	1

Because the truth values in columns 5 and 7 differ (in rows 5 and 7), we must avoid writing a compound statement such as $p \vee q \wedge r$. Without parentheses to indicate which of the connectives $\vee$ and $\wedge$ should be applied first, we have no idea whether we are dealing with $p \vee (q \wedge r)$ or $(p \vee q) \wedge r$.

Our last example for this section illustrates two special types of statements.

EXAMPLE 2.6

The results in columns 4 and 7 of Table 2.5 reveal that the statement $p \rightarrow (p \vee q)$ is true and that the statement $p \wedge (\neg p \wedge q)$ is false for all truth value assignments for the component statements p, q.

Table 2.5

p	q	$p \vee q$	$p \rightarrow (p \vee q)$	$\neg p$	$\neg p \wedge q$	$p \wedge (\neg p \wedge q)$
0	0	0	1	1	0	0
0	1	1	1	1	1	0
1	0	1	1	0	0	0
1	1	1	1	0	0	0

Definition 2.1

A compound statement is called a *tautology* if it is true for all truth value assignments for its component statements. If a compound statement is false for all such assignments, then it is called a *contradiction*.

Throughout this chapter we shall use the symbol T_0 to denote any tautology and the symbol F_0 to denote any contradiction.

We can use the ideas of tautology and implication to describe what we mean by a valid argument. This will be of primary interest to us in Section 2.3, and it will help us develop needed skills for proving mathematical theorems. In general, an argument starts with a list of *given* statements called *premises* and a statement called the *conclusion* of the argument. We examine these premises, say $p_1, p_2, p_3, \ldots, p_n$, and try to show that the conclusion q follows logically from these given statements — that is, we try to show that if each of $p_1, p_2, p_3, \ldots, p_n$ is a true statement, then the statement q is also true. To do so one way is to examine the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n)^{\dagger} \rightarrow q,$$

where the hypothesis is the conjunction of the n premises. If any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then no matter what truth value q has, the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is true. Consequently, if we start with the premises $p_1, p_2, p_3, \ldots, p_n$ — each with truth value 1 — and find that under these circumstances q also has the value 1, then the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$$

is a *tautology* and we have a *valid argument*.

† At this point we have dealt only with the conjunction of two statements, so we must point out that the conjunction $p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n$ of n statements is true if and only if each p_i, $1 \leq i \leq n$, is true. We shall deal with this generalized conjunction in detail in Example 4.16 of Section 4.2.

EXERCISES 2.1

1. Determine whether each of the following sentences is a statement.

a) In 2003 George W. Bush was the president of the United States.

b) $x + 3$ is a positive integer.

c) Fifteen is an even number.

d) If Jennifer is late for the party, then her cousin Zachary will be quite angry.

e) What time is it?

f) As of June 30, 2003, Christine Marie Evert had won the French Open a record seven times.

2. Identify the primitive statements in Exercise 1.

3. Let p, q be primitive statements for which the implication $p \rightarrow q$ is false. Determine the truth values for each of the following.

a) $p \wedge q$ **b)** $\neg p \vee q$ **c)** $q \rightarrow p$ **d)** $\neg q \rightarrow \neg p$

4. Let p, q, r, s denote the following statements:

p: I finish writing my computer program before lunch.
q: I shall play tennis in the afternoon.
r: The sun is shining.
s: The humidity is low.

Write the following in symbolic form.

a) If the sun is shining, I shall play tennis this afternoon.

b) Finishing the writing of my computer program before lunch is necessary for my playing tennis this afternoon.

c) Low humidity and sunshine are sufficient for me to play tennis this afternoon.

5. Let p, q, r denote the following statements about a particular triangle ABC.

p: Triangle ABC is isosceles.
q: Triangle ABC is equilateral.
r: Triangle ABC is equiangular.

Translate each of the following into an English sentence.

a) $q \rightarrow p$ **b)** $\neg p \rightarrow \neg q$

c) $q \leftrightarrow r$ **d)** $p \wedge \neg q$

e) $r \rightarrow p$

6. Determine the truth value of each of the following implications.

a) If $3 + 4 = 12$, then $3 + 2 = 6$.

b) If $3 + 3 = 6$, then $3 + 4 = 9$.

c) If Thomas Jefferson was the third president of the United States, then $2 + 3 = 5$.

7. Rewrite each of the following statements as an implication in the **if-then** form.

a) Practicing her serve daily is a sufficient condition for Darci to have a good chance of winning the tennis tournament.

b) Fix my air conditioner or I won't pay the rent.

c) Mary will be allowed on Larry's motorcycle only if she wears her helmet.

8. Construct a truth table for each of the following compound statements, where p, q, r denote primitive statements.

a) $\neg(p \vee \neg q) \rightarrow \neg p$ **b)** $p \rightarrow (q \rightarrow r)$

c) $(p \rightarrow q) \rightarrow r$ **d)** $(p \rightarrow q) \rightarrow (q \rightarrow p)$

e) $[p \wedge (p \rightarrow q)] \rightarrow q$ **f)** $(p \wedge q) \rightarrow p$

g) $q \leftrightarrow (\neg p \vee \neg q)$

h) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

9. Which of the compound statements in Exercise 8 are tautologies?

10. Verify that $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology.

11. a) How many rows are needed for the truth table of the compound statement $(p \vee \neg q) \leftrightarrow [(\neg r \wedge s) \rightarrow t]$, where p, q, r, s, and t are primitive statements?

b) Let $p_1, p_2, \ldots, p_n$ denote n primitive statements. Let p be a compound statement that contains at least one occurrence each of p_i, for $1 \leq i \leq n$—and p contains no other primitive statement. How many rows are needed to construct the truth table for p?

12. Determine all truth value assignments, if any, for the primitive statements p, q, r, s, t that make each of the following compound statements false.

a) $[(p \wedge q) \wedge r] \rightarrow (s \vee t)$

b) $[p \wedge (q \wedge r)] \rightarrow (s \veebar t)$

13. If statement q has the truth value 1, determine all truth value assignments for the primitive statements, p, r, and s for which the truth value of the statement

$$(q \rightarrow [(\neg p \vee r) \wedge \neg s]) \wedge [\neg s \rightarrow (\neg r \wedge q)]$$

is 1.

14. At the start of a program (written in pseudocode) the integer variable n is assigned the value 7. Determine the value of n after each of the following *successive* statements is encountered during the execution of this program. [Here the value of n following the execution of the statement in part (a) becomes the value of n for the statement in part (b), and so on, through the statement in part (d). For positive integers a, b, $\lfloor a/b \rfloor$ returns the integer part of the quotient—for example, $\lfloor 6/2 \rfloor = 3$, $\lfloor 7/2 \rfloor = 3$, $\lfloor 2/5 \rfloor = 0$, and $\lfloor 8/3 \rfloor = 2$.]

a) `if n > 5 then n := n + 2`

b) **if** ((*n* + 2 = 8) **or** (*n* - 3 = 6)) **then**
n := 2 * *n* + 1

c) **if** ((*n* - 3 = 16) **and** (⌊*n*/6⌋ = 1)) **then**
n := *n* + 3

d) **if** ((*n* ≠ 21) **and** (*n* - 7 = 15)) **then**
n := *n* - 4

15. The integer variables m and n are assigned the values 3 and 8, respectively, during the execution of a program (written in pseudocode). Each of the following *successive* statements is then encountered during program execution. [Here the values of m, n following the execution of the statement in part (a) become the values of m, n for the statement in part (b), and so on, through the statement in part (e).] What are the values of m, n after each of these statements is encountered?

a) **if** *n* - *m* = 5 **then** *n* := *n* - 2

b) **if** ((2 * *m* = *n*) **and** (⌊*n*/4⌋ = 1)) **then**
n := 4 * *m* - 3

c) **if** ((*n* < 8) **or** (⌊*m*/2⌋= 2)) **then** *n* := 2 * *m*
else *m* := 2 * n

d) **if** ((*m* < 20) **and** (⌊*n*/6⌋ = 1)) **then**
m := *m* - *n* - 5

e) **if** ((*n* = 2 * *m*) **or** (⌊*n*/2⌋ = 5)) **then**
m := *m* + 2

16. In the following program segment i, j, m, and n are integer variables. The values of m and n are supplied by the user earlier in the execution of the total program.

```
for i := 1 to m do
  for j := 1 to n do
    if i ≠ j then
      print i + j
```

How many times is the **print** statement in the segment executed when (a) $m = 10$, $n = 10$; (b) $m = 20$, $n = 20$; (c) $m = 10$, $n = 20$; (d) $m = 20$, $n = 10$?

17. After baking a pie for the two nieces and two nephews who are visiting her, Aunt Nellie leaves the pie on her kitchen table to cool. Then she drives to the mall to close her boutique for the day. Upon her return she finds that someone has eaten one-quarter of the pie. Since no one was in her house that day—except for the four visitors—Aunt Nellie questions each niece and nephew about who ate the piece of pie. The four "suspects" tell her the following:

Charles:	Kelly ate the piece of pie.
Dawn:	I did not eat the piece of pie.
Kelly:	Tyler ate the pie.
Tyler:	Kelly lied when she said I ate the pie.

If only one of these four statements is true and only one of the four committed this heinous crime, who is the vile culprit that Aunt Nellie will have to punish severely?

2.2 Logical Equivalence: The Laws of Logic

In all areas of mathematics we need to know when the entities we are studying are equal or essentially the same. For example, in arithmetic and algebra we know that two nonzero real numbers are equal when they have the same magnitude and algebraic sign. Hence, for two nonzero real numbers x, y, we have $x = y$ if $|x| = |y|$ and $xy > 0$, and conversely (that is, if $x = y$, then $|x| = |y|$ and $xy > 0$). When we deal with triangles in geometry, the notion of congruence arises. Here triangle ABC and triangle DEF are congruent if, for instance, they have equal corresponding sides—that is, the length of side AB = the length of side DE, the length of side BC = the length of side EF, and the length of side CA = the length of side FD.

Our study of logic is often referred to as the *algebra of propositions* (as opposed to the algebra of real numbers). In this algebra we shall use the truth tables of the statements, or propositions, to develop an idea of when two such entities are essentially the same. We begin with an example.

EXAMPLE 2.7

For primitive statements p and q, Table 2.6 provides the truth tables for the compound statements $\neg p \vee q$ and $p \rightarrow q$. Here we see that the corresponding truth tables for the two statements $\neg p \vee q$ and $p \rightarrow q$ are exactly the same.

Table 2.6

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

This situation leads us to the following idea.

Definition 2.2 Two statements s_1, s_2 are said to be *logically equivalent*, and we write $s_1 \Longleftrightarrow s_2$, when the statement s_1 is true (respectively, false) if and only if the statement s_2 is true (respectively, false).

Note that when $s_1 \Longleftrightarrow s_2$ the statements s_1 and s_2 provide the same truth tables because s_1, s_2 have the same truth values for *all* choices of truth values for their primitive components.

As a result of this concept we see that we can express the connective for the implication (of primitive statements) in terms of negation and disjunction — that is, $(p \rightarrow q) \Longleftrightarrow \neg p \vee q$. In the same manner, from the result in Table 2.7 we have $(p \leftrightarrow q) \Longleftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$, and this helps validate the use of the term *biconditional*. Using the logical equivalence from Table 2.6, we find that we can also write $(p \leftrightarrow q) \Longleftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$. Consequently, if we so choose, we can eliminate the connectives $\rightarrow$ and $\leftrightarrow$ from compound statements.

Table 2.7

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Examining Table 2.8, we find that negation, along with the connectives $\wedge$ and $\vee$, are all we need to replace the *exclusive or* connective, $\veebar$. In fact, we may even eliminate either $\wedge$ or $\vee$. However, for the related applications we want to study later in the text, we shall need both $\wedge$ and $\vee$ as well as negation.

Table 2.8

p	q	$p \veebar q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$	$(p \vee q) \wedge \neg(p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

We now use the idea of logical equivalence to examine some of the important properties that hold for the algebra of propositions.

For all real numbers a, b, we know that $-(a+b) = (-a) + (-b)$. Is there a comparable result for primitive statements p, q?

EXAMPLE 2.8

In Table 2.9 we have constructed the truth tables for the statements $\neg(p \wedge q)$, $\neg p \vee \neg q$, $\neg(p \vee q)$, and $\neg p \wedge \neg q$, where p, q are primitive statements. Columns 4 and 7 reveal that $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$; columns 9 and 10 reveal that $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$. These results are known as *DeMorgan's Laws*. They are similar to the familiar law for real numbers,

$$-(a+b) = (-a) + (-b),$$

already noted, which shows the negative of a sum to be equal to the sum of the negatives. Here, however, a crucial difference emerges: The negation of the *conjunction* of two primitive statements p, q results in the *disjunction* of their negations $\neg p$, $\neg q$, whereas the negation of the *disjunction* of these same statements p, q is logically equivalent to the *conjunction* of their negations $\neg p$, $\neg q$.

Table 2.9

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
0	0	0	1	1	1	1	0	1	1
0	1	0	1	1	0	1	1	0	0
1	0	0	1	0	1	1	1	0	0
1	1	1	0	0	0	0	1	0	0

Although p, q were primitive statements in the preceding example we shall soon learn that DeMorgan's Laws hold for any two arbitrary statements.

In the arithmetic of real numbers, the operations of addition and multiplication are both involved in the principle called the Distributive Law of Multiplication over Addition: For all real numbers a, b, c,

$$a \times (b + c) = (a \times b) + (a \times c).$$

The next example shows that there is a similar law for primitive statements. There is also a second related law (for primitive statements) that has no counterpart in the arithmetic of real numbers.

EXAMPLE 2.9

Table 2.10 contains the truth tables for the statements $p \wedge (q \vee r)$, $(p \wedge q) \vee (p \wedge r)$, $p \vee (q \wedge r)$, and $(p \vee q) \wedge (p \vee r)$. From the table it follows that for all primitive statements p, q, and r,

$p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$ The Distributive Law of $\wedge$ over $\vee$

$p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$ The Distributive Law of $\vee$ over $\wedge$

The second distributive law has no counterpart in the arithmetic of real numbers. That is, it is not true for all real numbers a, b, and c that the following holds: $a + (b \times c) = (a + b) \times (a + c)$. For $a = 2$, $b = 3$, and $c = 5$, for instance, $a + (b \times c) = 17$ but $(a + b) \times (a + c) = 35$.

Table 2.10

p	q	r	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r)$	$(p \vee q) \wedge (p \vee r)$
0	0	0	0	0	0	0
0	0	1	0	0	0	0
0	1	0	0	0	0	0
0	1	1	0	0	1	1
1	0	0	0	0	1	1
1	0	1	1	1	1	1
1	1	0	1	1	1	1
1	1	1	1	1	1	1

Before going any further, we note that, in general, if s_1, s_2 are statements and $s_1 \leftrightarrow s_2$ is a tautology, then s_1, s_2 must have the same corresponding truth values (that is, for each assignment of truth values to the primitive statements in s_1 and s_2, s_1 is true if and only if s_2 is true and s_1 is false if and only if s_2 is false) and $s_1 \Longleftrightarrow s_2$. When s_1 and s_2 are logically equivalent statements (that is, $s_1 \Longleftrightarrow s_2$), then the compound statement $s_1 \leftrightarrow s_2$ is a tautology. Under these circumstances it is also true that $\neg s_1 \Longleftrightarrow \neg s_2$, and $\neg s_1 \leftrightarrow \neg s_2$ is a tautology.

If s_1, s_2, and s_3 are statements where $s_1 \Longleftrightarrow s_2$ and $s_2 \Longleftrightarrow s_3$ then $s_1 \Longleftrightarrow s_3$. When two statements s_1 and s_2 are not logically equivalent, we may write $s_1 \not\Leftrightarrow s_2$ to designate this situation.

Using the concepts of logical equivalence, tautology, and contradiction, we state the following list of laws for the algebra of propositions.

The Laws of Logic

For any primitive statements p, q, r, any tautology T_0, and any contradiction F_0,

1) $\neg\neg p \Longleftrightarrow p$	Law of *Double Negation*
2) $\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q$ $\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q$	*DeMorgan's* Laws
3) $p \vee q \Longleftrightarrow q \vee p$ $p \wedge q \Longleftrightarrow q \wedge p$	*Commutative* Laws
4) $p \vee (q \vee r) \Longleftrightarrow (p \vee q) \vee r$[†] $p \wedge (q \wedge r) \Longleftrightarrow (p \wedge q) \wedge r$	*Associative* Laws
5) $p \vee (q \wedge r) \Longleftrightarrow (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \Longleftrightarrow (p \wedge q) \vee (p \wedge r)$	*Distributive* Laws
6) $p \vee p \Longleftrightarrow p$ $p \wedge p \Longleftrightarrow p$	*Idempotent* Laws
7) $p \vee F_0 \Longleftrightarrow p$ $p \wedge T_0 \Longleftrightarrow p$	*Identity* Laws

[†] We note that because of the Associative Laws, there is no ambiguity in statements of the form $p \vee q \vee r$ or $p \wedge q \wedge r$.

8) $p \vee \neg p \Leftrightarrow T_0$ — *Inverse* Laws
$p \wedge \neg p \Leftrightarrow F_0$

9) $p \vee T_0 \Leftrightarrow T_0$ — *Domination* Laws
$p \wedge F_0 \Leftrightarrow F_0$

10) $p \vee (p \wedge q) \Leftrightarrow p$ — *Absorption* Laws
$p \wedge (p \vee q) \Leftrightarrow p$

We now turn our attention to proving all of these properties. In so doing we realize that we could simply construct the truth tables and compare the results for the corresponding truth values in each case — as we did in Examples 2.8 and 2.9. However, before we start writing, let us take one more look at this list of 19 laws, which, aside from the Law of Double Negation, fall naturally into pairs. This pairing idea will help us after we examine the following concept.

Definition 2.3 Let s be a statement. If s contains no logical connectives other than $\wedge$ and $\vee$, then the *dual* of s, denoted s^d, is the statement obtained from s by replacing each occurrence of $\wedge$ and $\vee$ by $\vee$ and $\wedge$, respectively, and each occurrence of T_0 and F_0 by F_0 and T_0, respectively.

If p is any primitive statement, then p^d is the same as p — that is, the dual of a primitive statement is simply the same primitive statement. And $(\neg p)^d$ is the same as $\neg p$. The statements $p \vee \neg p$ and $p \wedge \neg p$ are duals of each other whenever p is primitive — and so are the statements $p \vee T_0$ and $p \wedge F_0$.

Given the primitive statements p, q, r and the compound statement

$$s: \quad (p \wedge \neg q) \vee (r \wedge T_0),$$

we find that the dual of s is

$$s^d: \quad (p \vee \neg q) \wedge (r \vee F_0).$$

(Note that $\neg q$ is unchanged as we go from s to s^d.)

We now state and use a theorem without proving it. However, in Chapter 15 we shall justify the result that appears here.

THEOREM 2.1 *The Principle of Duality.* Let s and t be statements that contain no logical connectives other than $\wedge$ and $\vee$. If $s \Leftrightarrow t$, then $s^d \Leftrightarrow t^d$.

As a result, laws 2 through 10 in our list can be established by proving one of the laws in each pair and then invoking this principle.

We also find that it is possible to derive many other logical equivalences. For example, if q, r, s are primitive statements, the results in columns 5 and 7 of Table 2.11 show us that

$$(r \wedge s) \rightarrow q \Leftrightarrow \neg(r \wedge s) \vee q$$

or that $[(r \wedge s) \rightarrow q] \leftrightarrow [\neg(r \wedge s) \vee q]$ is a tautology. However, instead of always constructing more (and, unfortunately, larger) truth tables it might be a good idea to recall from Example 2.7 that for primitive statements p, q, the compound statement

$$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$$

Table 2.11

q	r	s	$r \wedge s$	$(r \wedge s) \rightarrow q$	$\neg(r \wedge s)$	$\neg(r \wedge s) \vee q$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	1	1
0	1	1	1	0	0	0
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

is a tautology. If we were to *replace* each occurrence of this primitive statement p by the compound statement $r \wedge s$, then we would obtain the earlier tautology

$$[(r \wedge s) \rightarrow q] \leftrightarrow [\neg(r \wedge s) \vee q].$$

What has happened here illustrates the first of the following two *substitution rules*:

1) Suppose that the compound statement P is a tautology. If p is a *primitive* statement that appears in P and we replace *each* occurrence of p by the *same* statement q, then the resulting compound statement P_1 is also a tautology.

2) Let P be a compound statement where p is an arbitrary statement that appears in P, and let q be a statement such that $q \Longleftrightarrow p$. Suppose that in P we replace one or more occurrences of p by q. Then this replacement yields the compound statement P_1. Under these circumstances $P_1 \Longleftrightarrow P$.

These rules are further illustrated in the following two examples.

EXAMPLE 2.10

a) From the first of DeMorgan's Laws we know that for all primitive statements p, q, the compound statement

$$P: \quad \neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$$

is a tautology. When we replace each occurrence of p by $r \wedge s$, it follows from the first substitution rule that

$$P_1: \quad \neg[(r \wedge s) \vee q] \leftrightarrow [\neg(r \wedge s) \wedge \neg q]$$

is also a tautology. Extending this result one step further, we may replace each occurrence of q by $t \rightarrow u$. The same substitution rule now yields the tautology

$$P_2: \quad \neg[(r \wedge s) \vee (t \rightarrow u)] \leftrightarrow [\neg(r \wedge s) \wedge \neg(t \rightarrow u)],$$

and hence, by the remarks following shortly after Example 2.9, the logical equivalence

$$\neg[(r \wedge s) \vee (t \rightarrow u)] \Longleftrightarrow [\neg(r \wedge s) \wedge \neg(t \rightarrow u)].$$

b) For primitive statements p, q, we learn from the last column of Table 2.12 that the compound statement $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology. Consequently, if r, s, t, u are any statements, then by the first substitution rule we obtain the new tautology

$$[(r \rightarrow s) \wedge [(r \rightarrow s) \rightarrow (\neg t \vee u)]] \rightarrow (\neg t \vee u)$$

when we replace each occurrence of p by $r \rightarrow s$ and each occurrence of q by $\neg t \vee u$.

Table 2.12

p	q	$p \to q$	$p \wedge (p \to q)$	$[p \wedge (p \to q)] \to q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

EXAMPLE 2.11

a) For an application of the second substitution rule, let P denote the compound statement $(p \to q) \to r$. Because $(p \to q) \Longleftrightarrow \neg p \vee q$ (as shown in Example 2.7 and Table 2.6), if P_1 denotes the compound statement $(\neg p \vee q) \to r$, then $P_1 \Longleftrightarrow P$. (We also find that $[(p \to q) \to r] \leftrightarrow [(\neg p \vee q) \to r]$ is a tautology.)

b) Now let P represent the compound statement (actually a tautology) $p \to (p \vee q)$. Since $\neg\neg p \Longleftrightarrow p$, the compound statement P_1: $p \to (\neg\neg p \vee q)$ is derived from P by replacing *only the second occurrence* (but *not* the first occurrence) of p by $\neg\neg p$. The second substitution rule still implies that $P_1 \Longleftrightarrow P$. [Note that P_2: $\neg\neg p \to (\neg\neg p \vee q)$, derived by replacing *both* occurrences of p by $\neg\neg p$, is also logically equivalent to P.]

Our next example demonstrates how we can use the idea of logical equivalence together with the laws of logic and the substitution rules.

EXAMPLE 2.12

Negate and simplify the compound statement $(p \vee q) \to r$.

We organize our explanation as follows:

1) $(p \vee q) \to r \Longleftrightarrow \neg(p \vee q) \vee r$ [by the first substitution rule because $(s \to t) \leftrightarrow (\neg s \vee t)$ is a tautology for primitive statements s, t].
2) Negating the statements in step (1), we have $\neg[(p \vee q) \to r] \Longleftrightarrow \neg[\neg(p \vee q) \vee r]$.
3) From the first of DeMorgan's Laws and the first substitution rule, $\neg[\neg(p \vee q) \vee r] \Longleftrightarrow \neg\neg(p \vee q) \wedge \neg r$.
4) The Law of Double Negation and the second substitution rule now gives us $\neg\neg(p \vee q) \wedge \neg r \Longleftrightarrow (p \vee q) \wedge \neg r$.

From steps (1) through (4) we have $\neg[(p \vee q) \to r] \Longleftrightarrow (p \vee q) \wedge \neg r$.

When we wanted to write the negation of an implication, as in Example 2.12, we found that the concept of logical equivalence played a key role — in conjunction with the laws of logic and the substitution rules. This idea is important enough to warrant a second look.

EXAMPLE 2.13

Let p, q denote the primitive statements

p: Joan goes to Lake George. q: Mary pays for Joan's shopping spree.

and consider the implication

$p \to q$: If Joan goes to Lake George, then Mary will pay for Joan's shopping spree.

Here we want to write the negation of $p \to q$ in a way other than simply $\neg(p \to q)$. We want to avoid writing the negation as "It is not the case that if Joan goes to Lake George, then Mary will pay for Joan's shopping spree."

To accomplish this we consider the following. Since $p \to q \Longleftrightarrow \neg p \vee q$, it follows that $\neg(p \to q) \Longleftrightarrow \neg(\neg p \vee q)$. Then by DeMorgan's Law we have $\neg(\neg p \vee q) \Longleftrightarrow \neg\neg p \wedge \neg q$, and from the Law of Double Negation and the second substitution rule it follows that $\neg\neg p \wedge \neg q \Longleftrightarrow p \wedge \neg q$. Consequently,

$$\neg(p \to q) \Longleftrightarrow \neg(\neg p \vee q) \Longleftrightarrow \neg\neg p \wedge \neg q \Longleftrightarrow p \wedge \neg q,$$

and we may write the negation of $p \to q$ in this case as

$\neg(p \to q)$: Joan goes to Lake George, but Mary does not pay for Joan's shopping spree.

(*Note*: The negation of an if-then statement does *not* begin with the word *if*. It is *not* another *implication*.)

EXAMPLE 2.14

In Definition 2.3 the dual s^d of a statement s was defined only for statements involving negation and the basic connectives $\wedge$ and $\vee$. How does one determine the dual of a statement such as s: $p \to q$, where p, q are primitive?

Because $(p \to q) \Longleftrightarrow \neg p \vee q$, s^d is logically equivalent to the statement $(\neg p \vee q)^d$, which is $\neg p \wedge q$.

The implication $p \to q$ and certain statements related to it are now examined in the following example.

EXAMPLE 2.15

Table 2.13 gives the truth tables for the statements $p \to q$, $\neg q \to \neg p$, $q \to p$, and $\neg p \to \neg q$. The third and fourth columns of the table reveal that

$$(p \to q) \Longleftrightarrow (\neg q \to \neg p).$$

Table 2.13

p	q	$p \to q$	$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$
0	0	1	1	1	1
0	1	1	1	0	0
1	0	0	0	1	1
1	1	1	1	1	1

The statement $\neg q \to \neg p$ is called the *contrapositive* of the implication $p \to q$. Columns 5 and 6 of the table show that

$$(q \to p) \Longleftrightarrow (\neg p \to \neg q).$$

The statement $q \to p$ is called the *converse* of $p \to q$; $\neg p \to \neg q$ is called the *inverse* of $p \to q$. We also see from Table 2.13 that

$$(p \to q) \not\Longleftrightarrow (q \to p) \quad \text{and} \quad (\neg p \to \neg q) \not\Longleftrightarrow (\neg q \to \neg p).$$

Consequently, we must keep the implication and its converse straight. The fact that a certain implication $p \to q$ is true (in particular, as in row 2 of the table) does *not* require that the converse $q \rightarrow p$ also be true. However, it does necessitate the truth of the contrapositive $\neg q \rightarrow \neg p$.

Let us consider a specific example where p, q represent the statements

p: Jeff is concerned about his cholesterol (HDL and LDL) levels.

q: Jeff walks at least two miles three times a week.

Then we obtain

- (The implication: $p \rightarrow q$). If Jeff is concerned about his cholesterol levels, then he will walk at least two miles three times a week.
- (The contrapositive: $\neg q \rightarrow \neg p$). If Jeff does not walk at least two miles three times a week, then he is not concerned about his cholesterol levels.
- (The converse: $q \rightarrow p$). If Jeff walks at least two miles three times a week, then he is concerned about his cholesterol levels.
- (The inverse: $\neg p \rightarrow \neg q$). If Jeff is not concerned about his cholesterol levels, then he will not walk at least two miles three times a week.

If p is true and q is false, then the implication $p \rightarrow q$ and the contrapositive $\neg q \rightarrow \neg p$ are false, while the converse $q \rightarrow p$ and the inverse $\neg p \rightarrow \neg q$ are true. For the case where p is false and q is true, the implication $p \rightarrow q$ and the contrapositive $\neg q \rightarrow \neg p$ are now true, while the converse $q \rightarrow p$ and the inverse $\neg p \rightarrow \neg q$ are false. When p, q are both true or both false, then the implication is true, as are the contrapositive, converse, and inverse.

We turn now to two examples involving the simplification of compound statements. For simplicity, we shall list the major laws of logic being used, but we shall not mention any applications of our two substitution rules.

EXAMPLE 2.16

For primitive statements p, q, is there any simpler way to express the compound statement $(p \vee q) \wedge \neg(\neg p \wedge q)$ — that is, can we find a simpler statement that is logically equivalent to the one given?

Here one finds that

	Reasons
$(p \vee q) \wedge \neg(\neg p \wedge q)$	
$\Longleftrightarrow (p \vee q) \wedge (\neg\neg p \vee \neg q)$	DeMorgan's Law
$\Longleftrightarrow (p \vee q) \wedge (p \vee \neg q)$	Law of Double Negation
$\Longleftrightarrow (p \vee (q \wedge \neg q)$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee F_0$	Inverse Law
$\Longleftrightarrow p$	Identity Law

Consequently, we see that

$$(p \vee q) \wedge \neg(\neg p \wedge q) \Longleftrightarrow p,$$

so we can express the given compound statement by the simpler logically equivalent statement p.

EXAMPLE 2.17

Consider the compound statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

where p, q, r are primitive statements. This statement contains four occurrences of primitive statements, three negation symbols, and three connectives.

From the laws of logic it follows that

	Reasons
$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$	
$\Longleftrightarrow \neg\neg[(p \vee q) \wedge r] \wedge \neg\neg q$	DeMorgan's Law
$\Longleftrightarrow [(p \vee q) \wedge r] \wedge q$	Law of Double Negation
$\Longleftrightarrow (p \vee q) \wedge (r \wedge q)$	Associative Law of $\wedge$
$\Longleftrightarrow (p \vee q) \wedge (q \wedge r)$	Commutative Law of $\wedge$
$\Longleftrightarrow [(p \vee q) \wedge q] \wedge r$	Associative Law of $\wedge$
$\Longleftrightarrow q \wedge r$	Absorption Law (as well as the Commutative Laws for $\wedge$ and $\vee$)

Consequently, the original statement

$$\neg[\neg[(p \vee q) \wedge r] \vee \neg q]$$

is logically equivalent to the much simpler statement

$$q \wedge r,$$

where we find only two primitive statements, no negation symbols, and only one connective.

Note further that from Example 2.7 we have

$$\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Longleftrightarrow \neg[\neg[(p \vee q) \wedge r] \vee \neg q],$$

so it follows that

$$\neg[[(p \vee q) \wedge r] \rightarrow \neg q] \Longleftrightarrow q \wedge r.$$

We close this section with an application on how the ideas in Examples 2.16 and 2.17 can be used in simplifying switching networks.

EXAMPLE 2.18

A switching network is made up of wires and switches connecting two terminals T_1 and T_2. In such a network, each switch is either open (0), so that no current flows through it, or closed (1), so that current does flow through it.

In Fig. 2.1(a) we have a network with one switch. Each of parts (b) and (c) contains two (independent) switches.

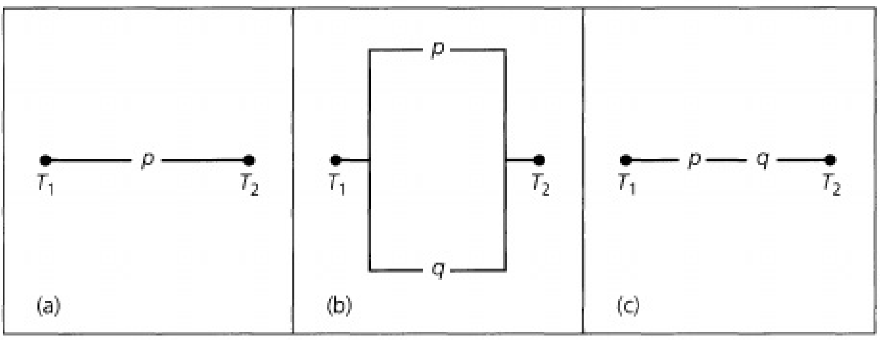

Figure 2.1

For the network in part (b), current flows from T_1 to T_2 if either of the switches p, q is closed. We call this a *parallel* network and represent it by $p \vee q$. The network in part (c) requires that each of the switches p, q be closed in order for current to flow from T_1 to T_2. Here the switches are in *series*; this network is represented by $p \wedge q$.

The switches in a network need not act independently of each other. Consider the network shown in Fig. 2.2(a). Here the switches labeled t and $\neg t$ are not independent. We have coupled these two switches so that t is open (closed) if and only if $\neg t$ is simultaneously closed (open). The same is true for the switches at q, $\neg q$. (Also, for example, the three switches labeled p are not independent.)

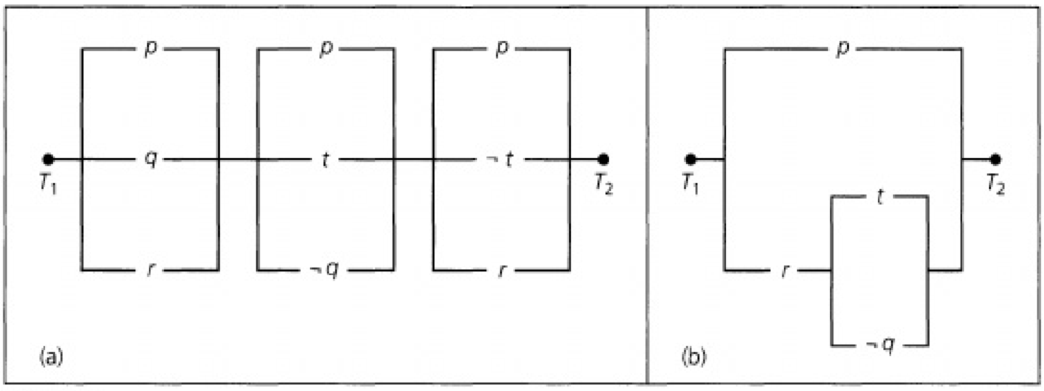

Figure 2.2

This network is represented by the statement $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$. Using the laws of logic, we may simplify this statement as follows.

	Reasons
$(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r)$	
$\Longleftrightarrow p \vee [(q \vee r) \wedge (t \vee \neg q) \wedge (\neg t \vee r)]$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee [(q \vee r) \wedge (\neg t \vee r) \wedge (t \vee \neg q)]$	Commutative Law of $\wedge$
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (t \vee \neg q)]$	Distributive Law of $\vee$ over $\wedge$
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge (\neg\neg t \vee \neg q)]$	Law of Double Negation
$\Longleftrightarrow p \vee [((q \wedge \neg t) \vee r) \wedge \neg(\neg t \wedge q)]$	DeMorgan's Law
$\Longleftrightarrow p \vee [\neg(\neg t \wedge q) \wedge ((\neg t \wedge q) \vee r)]$	Commutative Law of $\wedge$ (twice)
$\Longleftrightarrow p \vee [(\neg(\neg t \wedge q) \wedge (\neg t \wedge q)) \vee (\neg(\neg t \wedge q) \wedge r)]$	Distributive Law of $\wedge$ over $\vee$
$\Longleftrightarrow p \vee [F_0 \vee (\neg(\neg t \wedge q) \wedge r)]$	$\neg s \wedge s \Longleftrightarrow F_0$, for any statement s
$\Longleftrightarrow p \vee [(\neg(\neg t \wedge q)) \wedge r]$	F_0 is the identity for $\vee$
$\Longleftrightarrow p \vee [r \wedge \neg(\neg t \wedge q)]$	Commutative Law of $\wedge$
$\Longleftrightarrow p \vee [r \wedge (t \vee \neg q)]$	DeMorgan's Law and the Law of Double Negation

Hence $(p \vee q \vee r) \wedge (p \vee t \vee \neg q) \wedge (p \vee \neg t \vee r) \Longleftrightarrow p \vee [r \wedge (t \vee \neg q)]$, and the network shown in Fig. 2.2(b) is equivalent to the original network in the sense that current flows from T_1 to T_2 in network (a) exactly when it does so in network (b). But network (b) has only four switches, five fewer than network (a).

EXERCISES 2.2

1. Let p, q, r denote primitive statements.

a) Use truth tables to verify the following logical equivalences.

i) $p \rightarrow (q \wedge r) \Longleftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$
ii) $[(p \vee q) \rightarrow r] \Longleftrightarrow [(p \rightarrow r) \wedge (q \rightarrow r)]$
iii) $[p \rightarrow (q \vee r)] \Longleftrightarrow [\neg r \rightarrow (p \rightarrow q)]$

b) Use the substitution rules to show that

$$[p \rightarrow (q \vee r)] \Longleftrightarrow [(p \wedge \neg q) \rightarrow r].$$

2. Verify the first Absorption Law by means of a truth table.

3. Use the substitution rules to verify that each of the following is a tautology. (Here p, q, and r are primitive statements.)

a) $[p \vee (q \wedge r)] \vee \neg[p \vee (q \wedge r)]$

b) $[(p \vee q) \rightarrow r] \leftrightarrow [\neg r \rightarrow \neg(p \vee q)]$

4. For primitive statements p, q, r, and s, simplify the compound statement

$$[[[(p \wedge q) \wedge r] \vee [(p \wedge q) \wedge \neg r]] \vee \neg q] \rightarrow s.$$

5. Negate and express each of the following statements in smooth English.

a) Kelsey will get a good education if she puts her studies before her interest in cheerleading.

b) Norma is doing her homework, and Karen is practicing her piano lessons.

c) If Harold passes his C++ course and finishes his data structures project, then he will graduate at the end of the semester.

6. Negate each of the following and simplify the resulting statement.

a) $p \wedge (q \vee r) \wedge (\neg p \vee \neg q \vee r)$

b) $(p \wedge q) \rightarrow r$

c) $p \rightarrow (\neg q \wedge r)$

d) $p \vee q \vee (\neg p \wedge \neg q \wedge r)$

7. a) If p, q are primitive statements, prove that

$$(\neg p \vee q) \wedge (p \wedge (p \wedge q)) \Longleftrightarrow (p \wedge q).$$

b) Write the dual of the logical equivalence in part (a).

8. Write the dual for (a) $q \rightarrow p$, (b) $p \rightarrow (q \wedge r)$, (c) $p \leftrightarrow q$, and (d) $p \veebar q$, where p, q, and r are primitive statements.

9. Write the converse, inverse, and contrapositive of each of the following implications. For each implication, determine its truth value as well as the truth values of its corresponding converse, inverse, and contrapositive.

a) If $0 + 0 = 0$, then $1 + 1 = 1$.

b) If $-1 < 3$ and $3 + 7 = 10$, then $\sin\left(\frac{3\pi}{2}\right) = -1$.

10. Determine whether each of the following is true or false. Here p, q are arbitrary statements.

a) An equivalent way to express the converse of "p is sufficient for q" is "p is necessary for q."

b) An equivalent way to express the inverse of "p is necessary for q" is "$\neg q$ is sufficient for $\neg p$."

c) An equivalent way to express the contrapositive of "p is necessary for q" is "$\neg q$ is necessary for $\neg p$."

11. Let p, q, and r denote primitive statements. Find a form of the contrapositive of $p \rightarrow (q \rightarrow r)$ with (a) only one occurrence of the connective $\rightarrow$; (b) no occurrences of the connective $\rightarrow$.

12. Show that for primitive statements p, q,

$$p \veebar q \Longleftrightarrow [(p \wedge \neg q) \vee (\neg p \wedge q)] \Longleftrightarrow \neg(p \leftrightarrow q).$$

13. Verify that $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)] \Longleftrightarrow [(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$, for primitive statements p, q, and r.

14. For primitive statements p, q,

a) verify that $p \rightarrow [q \rightarrow (p \wedge q)]$ is a tautology.

b) verify that $(p \vee q) \rightarrow [q \rightarrow q]$ is a tautology by using the result from part (a) along with the substitution rules and the laws of logic.

c) is $(p \vee q) \rightarrow [q \rightarrow (p \wedge q)]$ a tautology?

15. Define the connective "Nand" or "Not ... and ..." by $(p \uparrow q) \Longleftrightarrow \neg(p \wedge q)$, for any statements p, q. Represent the following using only this connective.

a) $\neg p$ **b)** $p \vee q$ **c)** $p \wedge q$
d) $p \rightarrow q$ **e)** $p \leftrightarrow q$

16. The connective "Nor" or "Not ... or ..." is defined for any statements p, q by $(p \downarrow q) \Longleftrightarrow \neg(p \vee q)$. Represent the statements in parts (a) through (e) of Exercise 15, using only this connective.

17. For any statements p, q, prove that

a) $\neg(p \downarrow q) \Longleftrightarrow (\neg p \uparrow \neg q)$

b) $\neg(p \uparrow q) \Longleftrightarrow (\neg p \downarrow \neg q)$

18. Give the reasons for each step in the following simplifications of compound statements.

a) $[(p \vee q) \wedge (p \vee \neg q)] \vee q$ **Reasons**
$\Longleftrightarrow [p \vee (q \wedge \neg q)] \vee q$
$\Longleftrightarrow (p \vee F_0) \vee q$
$\Longleftrightarrow p \vee q$

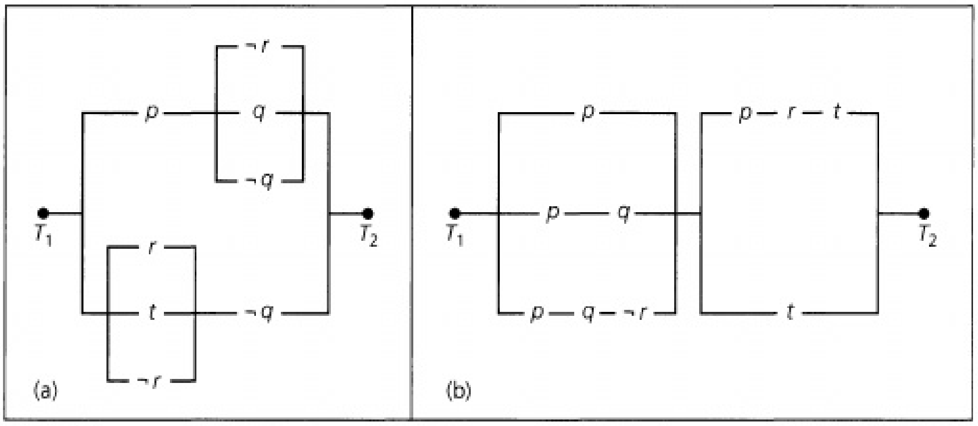

Figure 2.3

b) $(p \rightarrow q) \wedge [\neg q \wedge (r \vee \neg q)]$ **Reasons**

$\Leftrightarrow (p \rightarrow q) \wedge \neg q$

$\Leftrightarrow (\neg p \vee q) \wedge \neg q$

$\Leftrightarrow \neg q \wedge (\neg p \vee q)$

$\Leftrightarrow (\neg q \wedge \neg p) \vee (\neg q \wedge q)$

$\Leftrightarrow (\neg q \wedge \neg p) \vee F_0$

$\Leftrightarrow \neg q \wedge \neg p$

$\Leftrightarrow \neg(q \vee p)$

19. Provide the steps and reasons, as in Exercise 18, to establish the following logical equivalences.

a) $p \vee [p \wedge (p \vee q)] \Leftrightarrow p$

b) $p \vee q \vee (\neg p \wedge \neg q \wedge r) \Leftrightarrow p \vee q \vee r$

c) $[(\neg p \vee \neg q) \rightarrow (p \wedge q \wedge r)] \Leftrightarrow p \wedge q$

20. Simplify each of the networks shown in Fig. 2.3.

2.3 Logical Implication: Rules of Inference

At the end of Section 2.1 we mentioned the notion of a valid argument. Now we will begin a formal study of what we shall mean by an argument and when such an argument is valid. This in turn will help us when we investigate how to prove theorems throughout the text.

We start by considering the general form of an argument, one we wish to show is valid. So let us consider the implication

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q.$$

Here n is a positive integer, the statements $p_1, p_2, p_3, \ldots, p_n$ are called the *premises* of the argument, and the statement q is the *conclusion* for the argument.

The preceding argument is called *valid* if whenever each of the premises $p_1, p_2, p_3, \ldots, p_n$ is true, then the conclusion q is likewise true. [Note that if any one of $p_1, p_2, p_3, \ldots, p_n$ is false, then the hypothesis $p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n$ is false and the implication $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is automatically true, regardless of the truth value of q.] Consequently, one way to establish the validity of a given argument is to show that the statement $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology.

The following examples illustrate this particular approach.

EXAMPLE 2.19

Let p, q, r denote the primitive statements given as

p: Roger studies.

q: Roger plays racketball.

r: Roger passes discrete mathematics.

Now let p_1, p_2, p_3 denote the premises

p_1: If Roger studies, then he will pass discrete mathematics.

p_2: If Roger doesn't play racketball, then he'll study.

p_3: Roger failed discrete mathematics.

We want to determine whether the argument

$$(p_1 \land p_2 \land p_3) \rightarrow q$$

is valid. To do so, we rewrite p_1, p_2, p_3 as

$$p_1: \quad p \rightarrow r \qquad p_2: \quad \neg q \rightarrow p \qquad p_3: \quad \neg r$$

and examine the truth table for the implication

$$[(p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r] \rightarrow q$$

given in Table 2.14. Because the final column in Table 2.14 contains all 1's, the implication is a tautology. Hence we can say that $(p_1 \land p_2 \land p_3) \rightarrow q$ is a valid argument.

Table 2.14

			p_1	p_2	p_3	$(p_1 \land p_2 \land p_3) \rightarrow q$
p	q	r	$p \rightarrow r$	$\neg q \rightarrow p$	$\neg r$	$[(p \rightarrow r) \land (\neg q \rightarrow p) \land \neg r] \rightarrow q$
0	0	0	1	0	1	1
0	0	1	1	0	0	1
0	1	0	1	1	1	1
0	1	1	1	1	0	1
1	0	0	0	1	1	1
1	0	1	1	1	0	1
1	1	0	0	1	1	1
1	1	1	1	1	0	1

EXAMPLE 2.20

Let us now consider the truth table in Table 2.15. The results in the last column of this table show that for any primitive statements p, r, and s, the implication

$$[p \land ((p \land r) \rightarrow s)] \rightarrow (r \rightarrow s)$$

Table 2.15

p_1				p_2	q	$(p_1 \land p_2) \rightarrow q$
p	r	s	$p \land r$	$(p \land r) \rightarrow s$	$r \rightarrow s$	$[(p \land ((p \land r) \rightarrow s)] \rightarrow (r \rightarrow s)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	1
1	1	1	1	1	1	1

is a tautology. Consequently, for premises

$$p_1: \quad p \qquad p_2: \quad (p \wedge r) \rightarrow s$$

and conclusion q: $(r \rightarrow s)$, we know that $(p_1 \wedge p_2) \rightarrow q$ is a valid argument, and we may say that the truth of the conclusion q is *deduced* or *inferred* from the truth of the premises p_1, p_2.

The idea presented in the preceding two examples leads to the following.

Definition 2.4 If p, q are arbitrary statements such that $p \rightarrow q$ is a tautology, then we say that p *logically implies* q and we write $p \Rightarrow q$ to denote this situation.

When p, q are statements and $p \Rightarrow q$, the implication $p \rightarrow q$ is a tautology and we refer to $p \rightarrow q$ as a *logical implication*. Note that we can avoid dealing with the idea of a tautology here by saying that $p \Rightarrow q$ (that is, p logically implies q) if q is true whenever p is true.

In Example 2.6 we found that for primitive statements p, q, the implication $p \rightarrow (p \vee q)$ is a tautology. In this case, therefore, we can say that p logically implies $p \vee q$ and write $p \Rightarrow (p \vee q)$. Furthermore, because of the first substitution rule, we also find that $p \Rightarrow (p \vee q)$ for any statements p, q — that is, $p \rightarrow (p \vee q)$ is a tautology for any statements p, q, whether or not they are primitive statements.

Let p, q be arbitrary statements.

1) If $p \Longleftrightarrow q$, then the statement $p \leftrightarrow q$ is a tautology, so the statements p, q have the same (corresponding) truth values. Under these conditions the statements $p \rightarrow q$, $q \rightarrow p$ are tautologies, and we have $p \Rightarrow q$ and $q \Rightarrow p$.
2) Conversely, suppose that $p \Rightarrow q$ and $q \Rightarrow p$. The logical implication $p \rightarrow q$ tells us that we never have statement p with the truth value 1 and statement q with the truth value 0. But could we have q with the truth value 1 and p with the truth value 0? If this occurred, we could not have the logical implication $q \rightarrow p$. Therefore, when $p \Rightarrow q$ and $q \Rightarrow p$, the statements p, q have the same (corresponding) truth values and $p \Longleftrightarrow q$.

Finally, the notation $p \not\Rightarrow q$ is used to indicate that $p \rightarrow q$ is *not* a tautology — so the given implication (namely, $p \rightarrow q$) is *not* a logical implication.

EXAMPLE 2.21 From the results in Example 2.8 (Table 2.9) and the first substitution rule, we know that for statements p, q,

$$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q.$$

Consequently,

$$\neg(p \wedge q) \Rightarrow (\neg p \vee \neg q) \quad \text{and} \quad (\neg p \vee \neg q) \Rightarrow \neg(p \wedge q)$$

for all statements p, q. Alternatively, because each of the implications

$$\neg(p \wedge q) \rightarrow (\neg p \vee \neg q) \quad \text{and} \quad (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$$

is a tautology, we may also write

$$[\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)] \Longleftrightarrow T_0 \quad \text{and} \quad [(\neg p \vee \neg q) \rightarrow \neg(p \wedge q)] \Longleftrightarrow T_0.$$

Returning now to our study of techniques for establishing the validity of an argument, we must take a careful look at the size of Tables 2.14 and 2.15. Each table has eight rows. For Table 2.14 we were able to express the three premises p_1, p_2, and p_3, and the conclusion q, in terms of the three primitive statements p, q, and r. A similar situation arose for the argument we analyzed in Table 2.15, where we had only two premises. But if we were confronted, for example, with establishing whether

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \to \neg p$$

is a logical implication (or presents a valid argument), the needed table would require $2^5 = 32$ rows. As the number of premises gets larger and our truth tables grow to 64, 128, 256, or more rows, this first technique for establishing the validity of an argument rapidly loses its appeal.

Furthermore, looking at Table 2.14 once again, we realize that in order to establish whether

$$[(p \to r) \wedge (\neg q \to p) \wedge \neg r] \to q$$

is a valid argument, we need to consider only those rows of the table where each of the three premises $p \to r$, $\neg q \to p$, and $\neg r$ has the truth value 1. (Remember that if the hypothesis — consisting of the conjunction of all of the premises — is false, then the implication is true regardless of the truth value of the conclusion.) This happens only in the third row, so a good deal of Table 2.14 is not really necessary. (It is not always the case that only one row has all of the premises true. Note that in Table 2.15 we would be concerned with the results in rows 5, 6, and 8.)

Consequently, what these observations are telling us is that we can possibly eliminate a great deal of the effort put into constructing the truth tables in Table 2.14 and Table 2.15. And since we want to avoid even larger tables, we are persuaded to develop a list of techniques called *rules of inference* that will help us as follows:

1) Using these techniques will enable us to consider only the cases wherein all the premises are true. Hence we consider the conclusion only for those rows of a truth table wherein each premise has the truth value 1 — and we do *not* construct the truth table.

2) The rules of inference are fundamental in the development of a step-by-step validation of how the conclusion q logically follows from the premises $p_1, p_2, p_3, \ldots, p_n$ in an implication of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \to q.$$

Such a development will establish the validity of the given argument, for it will show how the truth of the conclusion can be deduced from the truth of the premises.

Each rule of inference arises from a logical implication. In some cases, the logical implication is stated without proof. (However, several of these proofs will be dealt with in the Section Exercises.)

Many rules of inference arise in the study of logic. We concentrate on those that we need to help us validate the arguments that arise in our study of logic. These rules will also help us later when we turn to methods for proving theorems throughout the remainder of the text. Table 2.19 (on p. 78) summarizes the rules we shall now start to investigate.

EXAMPLE 2.22

For a first example we consider the rule of inference called *Modus Ponens,* or the *Rule of Detachment.* (*Modus Ponens* comes from Latin and may be translated as "the method of affirming.") In symbolic form this rule is expressed by the logical implication

$$[p \land (p \rightarrow q)] \rightarrow q,$$

which is verified in Table 2.16, where we find that the fourth row is the only one where both of the premises p and $p \rightarrow q$ (and the conclusion q) are true.

Table 2.16

p	q	$p \rightarrow q$	$p \land (p \rightarrow q)$	$[p \land (p \rightarrow q)] \rightarrow q$
0	0	1	0	1
0	1	1	0	1
1	0	0	0	1
1	1	1	1	1

The actual rule will be written in the tabular form

$$\begin{array}{l} \quad p \\ \quad p \rightarrow q \\ \hline \therefore q \end{array}$$

where the three dots ($\therefore$) stand for the word "therefore," indicating that q is the conclusion for the premises p and $p \rightarrow q$, which appear above the horizontal line.

This rule arises when we argue that *if* (1) p is true, *and* (2) $p \rightarrow q$ is true (or $p \Rightarrow q$), *then* the conclusion q must also be true. (After all, if q were false and p were true, then we could not have $p \rightarrow q$ true.)

The following valid arguments show us how to apply the Rule of Detachment.

a) **1)** Lydia wins a ten-million-dollar lottery. $\quad p$
2) If Lydia wins a ten-million-dollar lottery, then Kay will quit her job. $\quad p \rightarrow q$
3) Therefore Kay will quit her job. $\quad \therefore q$

b) **1)** If Allison vacations in Paris, then she will have to win a scholarship. $\quad p \rightarrow q$
2) Allison is vacationing in Paris. $\quad p$
3) Therefore Allison won a scholarship. $\quad \therefore q$

Before closing the discussion on our first rule of inference let us make one final observation. The two examples in (a) and (b) might suggest that the valid argument $[p \land (p \rightarrow q)] \rightarrow q$ is appropriate only for primitive statements p, q. However, since $[p \land (p \rightarrow q)] \rightarrow q$ is a tautology for primitive statements p, q, it follows from the first substitution rule that (all occurrences of) p or q may be replaced by compound statements — and the resulting implication will also be a tautology. Consequently, if r, s, t, and u are primitive statements, then

$$\begin{array}{l} \quad r \lor s \\ \quad (r \lor s) \rightarrow (\neg t \land u) \\ \hline \therefore \neg t \land u \end{array}$$

is a valid argument, by the Rule of Detachment — just as $[(r \lor s) \land [(r \lor s) \rightarrow (\neg t \land u)]] \rightarrow (\neg t \land u)$ is a tautology.

A similar situation — in which we can apply the first substitution rule — occurs for each of the rules of inference we shall study. However, we shall not mention this so explicitly with these other rules of inference.

EXAMPLE 2.23

A second rule of inference is given by the logical implication

$$[(p \to q) \land (q \to r)] \to (p \to r),$$

where p, q, and r are any statements. In tabular form it is written

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore p \to r \end{array}$$

This rule, which is referred to as the *Law of the Syllogism,* arises in many arguments. For example, we may use it as follows:

1) If the integer 35244 is divisible by 396, then the integer 35244 is divisible by 66. $\qquad p \to q$
2) If the integer 35244 is divisible by 66, then the integer 35244 is divisible by 3. $\qquad q \to r$
3) Therefore, if the integer 35244 is divisible by 396, then the integer 35244 is divisible by 3. $\qquad \therefore p \to r$

The next example involves a slightly longer argument that uses the rules of inference developed in Examples 2.22 and 2.23. In fact, we find here that there may be more than one way to establish the validity of an argument.

EXAMPLE 2.24

Consider the following argument.

1) Rita is baking a cake.
2) If Rita is baking a cake, then she is not practicing her flute.
3) If Rita is not practicing her flute, then her father will not buy her a car.
4) Therefore Rita's father will not buy her a car.

Concentrating on the forms of the statements in the preceding argument, we may write the argument as

$$\begin{array}{l} p \\ p \to \neg q \\ \neg q \to \neg r \\ \hline \therefore \neg r \end{array} \qquad (*)$$

Now we need no longer worry about what the statements actually stand for. Our objective is to use the two rules of inference that we have studied so far in order to deduce the truth of the statement $\neg r$ from the truth of the three premises p, $p \to \neg q$, and $\neg q \to \neg r$.

We establish the validity of the argument as follows:

Steps	**Reasons**
1) $p \to \neg q$	Premise
2) $\neg q \to \neg r$	Premise
3) $p \to \neg r$	This follows from steps (1) and (2) and the Law of the Syllogism
4) p	Premise
5) $\therefore \neg r$	This follows from steps (4) and (3) and the Rule of Detachment

Before continuing with a third rule of inference we shall show that the argument presented at (*) can be validated in a second way. Here our "reasons" will be shortened to the form we shall use for the rest of the section. However, we shall always list whatever is needed to demonstrate how each step in an argument comes about, or follows, from prior steps.

A second way to validate the argument follows.

Steps	**Reasons**
1) p	Premise
2) $p \to \neg q$	Premise
3) $\neg q$	Steps (1) and (2) and the Rule of Detachment
4) $\neg q \to \neg r$	Premise
5) $\therefore \neg r$	Steps (3) and (4) and the Rule of Detachment

EXAMPLE 2.25

The rule of inference called *Modus Tollens* is given by

$$\begin{array}{r} p \to q \\ \neg q \\ \hline \therefore \neg p \end{array}$$

This follows from the logical implication $[(p \to q) \wedge \neg q] \to \neg p$. *Modus Tollens* comes from Latin and can be translated as "method of denying." This is appropriate because we deny the conclusion, q, so as to prove $\neg p$. (Note that we can also obtain this rule from the one for Modus Ponens by using the fact that $p \to q \Longleftrightarrow \neg q \to \neg p$.)

The following exemplifies the use of Modus Tollens is making a valid inference:

1) If Connie is elected president of Phi Delta sorority, then Helen will pledge that sorority. $p \to q$
2) Helen did not pledge Phi Delta sorority. $\neg q$
3) Therefore Connie was not elected president of Phi Delta sorority. $\therefore \neg p$

And now we shall use Modus Tollens to show that the following argument is valid (for primitive statements p, r, s, t, and u).

$$\begin{array}{r} p \to r \\ r \to s \\ t \vee \neg s \\ \neg t \vee u \\ \neg u \\ \hline \therefore \neg p \end{array}$$

Both Modus Tollens and the Law of the Syllogism come into play, along with the logical equivalence we developed in Example 2.7.

Steps	**Reasons**
1) $p \to r,\ r \to s$	Premises
2) $p \to s$	Step (1) and the Law of the Syllogism
3) $t \vee \neg s$	Premise
4) $\neg s \vee t$	Step (3) and the Commutative Law of $\vee$
5) $s \to t$	Step (4) and the fact that $\neg s \vee t \Longleftrightarrow s \to t$
6) $p \to t$	Steps (2) and (5) and the Law of the Syllogism
7) $\neg t \vee u$	Premise
8) $t \to u$	Step (7) and the fact that $\neg t \vee u \Longleftrightarrow t \to u$
9) $p \to u$	Steps (6) and (8) and the Law of the Syllogism
10) $\neg u$	Premise
11) $\therefore \neg p$	Steps (9) and (10) and Modus Tollens

Before continuing with another rule of inference let us summarize what we have just accomplished (and *not* accomplished). The preceding argument shows that

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \Rightarrow \neg p.$$

We have *not* used the laws of logic, as in Section 2.2, to express the statement

$$(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u$$

as a simpler logically equivalent statement. Note that

$$[(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u] \not\Leftrightarrow \neg p.$$

For when p has the truth value 0 and u has the truth value 1, the truth value of $\neg p$ is 1 while that of $\neg u$ and $(p \to r) \wedge (r \to s) \wedge (t \vee \neg s) \wedge (\neg t \vee u) \wedge \neg u$ is 0.

Let us once more examine a tabular form for each of the two related rules of inference, Modus Ponens and Modus Tollens.

Modus Ponens:
$$\begin{array}{l} p \to q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

Modus Tollens:
$$\begin{array}{l} p \to q \\ \underline{\neg q \qquad} \\ \therefore \neg p \end{array}$$

The reason we wish to do this is that there are other tabular forms that may arise — and these are similar in appearance but present *invalid* arguments — where each of the premises is true but the conclusion is false.

a) Consider the following argument:

1) If Margaret Thatcher is the president of the United States, then she is at least 35 years old. $p \to q$

2) Margaret Thatcher is at least 35 years old. $\underline{q}$

3) Therefore Margaret Thatcher is the president of the United States. $\therefore p$

Here we find that $[(p \to q) \wedge q] \to p$ is *not* a tautology. For if we consider the truth value assignments p: 0 and q: 1, then each of the premises $p \to q$ and q is true while the conclusion p is false. This *invalid* argument results from the *fallacy* (error in reasoning) where we try to argue by the converse — that is, while $[(p \to q) \wedge p] \Rightarrow q$, it is *not the case* that $[(p \to q) \wedge q] \Rightarrow p$.

b) A second argument where the conclusion doesn't necessarily follow from the premises may be given by:

1) If $2+3=6$, then $2+4=6$. $p \rightarrow q$
2) $2+3 \neq 6$. $\neg p$
3) Therefore $2+4 \neq 6$. $\therefore \neg q$

In this case we find that $[(p \rightarrow q) \wedge \neg p] \rightarrow \neg q$ is *not* a tautology. Once again the truth value assignments p: 0 and q: 1 show us that the premises $p \rightarrow q$ and $\neg p$ can both be true while the conclusion $\neg q$ is false. The fallacy behind this invalid argument arises from our attempt to argue by the inverse—for although $[(p \rightarrow q) \wedge \neg q] \Rightarrow \neg p$, it does *not* follow that $[(p \rightarrow q) \wedge \neg p] \Rightarrow \neg q$.

Before proceeding further we now mention a rather simple but important rule of inference.

EXAMPLE 2.26

The following rule of inference arises from the observation that if p, q are true statements, then $p \wedge q$ is a true statement.

Now suppose that statements p, q occur in the development of an argument. These statements may be (given) premises or results that are derived from premises and/or from results developed earlier in the argument. Then under these circumstances the two statements p, q can be combined into their conjunction $p \wedge q$, and this new statement can be used in later steps as the argument continues.

We call this rule the *Rule of Conjunction* and write it in tabular form as

$$\begin{array}{l} p \\ \underline{q} \\ \therefore p \wedge q \end{array}$$

As we proceed further with our study of rules of inference, we find another fairly simple but important rule.

EXAMPLE 2.27

The following rule of inference — one we may feel just illustrates good old common sense — is called the *Rule of Disjunctive Syllogism*. This rule comes about from the logical implication

$$[(p \vee q) \wedge \neg p] \rightarrow q,$$

which we can derive from Modus Ponens by observing that $p \vee q \Leftrightarrow \neg p \rightarrow q$.

In tabular form we write

$$\begin{array}{l} p \vee q \\ \underline{\neg p} \\ \therefore q \end{array}$$

This rule of inference arises when there are exactly two possibilities to consider and we are able to eliminate one of them as being true. Then the other possibility has to be true. The following illustrates one such application of this rule.

1) Bart's wallet is in his back pocket or it is on his desk. $p \vee q$
2) Bart's wallet is not in his back pocket. $\neg p$
3) Therefore Bart's wallet is on his desk. $\therefore q$

At this point we have examined five rules of inference. But before we try to validate any more arguments like the one (with 11 steps) in Example 2.25, we shall look at one more of these rules. This one underlies a method of proof that is sometimes confused with the contrapositive method (or proof) given in Modus Tollens. The confusion arises because both methods involve the negation of a statement. However, we will soon realize that these are two distinct methods. (Toward the end of Section 2.5 we shall compare and contrast these two methods once again.)

EXAMPLE 2.28

Let p denote an arbitrary statement, and F_0 a contradiction. The results in column 5 of Table 2.17 show that the implication $(\neg p \rightarrow F_0) \rightarrow p$ is a tautology, and this provides us with the rule of inference called the *Rule of Contradiction*. In tabular form this rule is written as

$$\frac{\neg p \rightarrow F_0}{\therefore p}$$

Table 2.17

p	$\neg p$	F_0	$\neg p \rightarrow F_0$	$(\neg p \rightarrow F_0) \rightarrow p$
1	0	0	1	1
0	1	0	0	1

This rule tells us that if p is a statement and $\neg p \rightarrow F_0$ is true, then $\neg p$ must be false because F_0 is false. So then we have p true.

The Rule of Contradiction is the basis of a method for establishing the validity of an argument — namely, the method of *Proof by Contradiction*, or *Reductio ad Absurdum*. The idea behind the method of Proof by Contradiction is to establish a statement (namely, the conclusion of an argument) by showing that, if this statement were false, then we would be able to deduce an impossible consequence. The use of this method arises in certain arguments which we shall now describe.

In general, when we want to establish the validity of the argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q,$$

we can establish the validity of the logically equivalent argument

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q) \rightarrow F_0.$$

[This follows from the tautology in column 7 of Table 2.18 and the first substitution rule — where we replace the primitive statement p by the statement $(p_1 \wedge p_2 \wedge \cdots \wedge p_n)^{\dagger}$.]

Table 2.18

p	q	F_0	$p \wedge \neg q$	$(p \wedge \neg q) \rightarrow F_0$	$p \rightarrow q$	$(p \rightarrow q) \leftrightarrow [(p \wedge \neg q) \rightarrow F_0]$
0	0	0	0	1	1	1
0	1	0	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

† In Section 4.2 we shall provide the reason why we know that for any statements $p_1, p_2, \ldots, p_n$, and q, it follows that $(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \wedge \neg q \Longleftrightarrow p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge \neg q$.

When we apply the method of Proof by Contradiction, we first assume that what we are trying to validate (or prove) is actually false. Then we use this assumption as an additional premise in order to produce a contradiction (or impossible situation) of the form $s \wedge \neg s$, for some statement s. Once we have derived this contradiction we may then conclude that the statement we were given was in fact true — and this validates the argument (or completes the proof).

We shall turn to the method of Proof by Contradiction when it is (or appears to be) easier to use $\neg q$ *in conjunction with* the premises $p_1, p_2, \ldots, p_n$ in order to deduce a contradiction than it is to deduce the conclusion q directly from the premises $p_1, p_2, \ldots, p_n$. The method of Proof by Contradiction will be used in some of the later examples for this section — namely, Examples 2.32 and 2.35. We shall also find it frequently reappearing in other chapters in the text.

Now that we have examined six rules of inference, we summarize these rules and introduce several others in Table 2.19 (on the following page).

The next five examples will present valid arguments. In so doing, these examples will show us how to apply the rules listed in Table 2.19 in conjunction with other results, such as the laws of logic.

EXAMPLE 2.29

Our first example demonstrates the validity of the argument

$$\begin{array}{l} p \to r \\ \neg p \to q \\ \underline{q \to s} \\ \therefore \neg r \to s \end{array}$$

Steps	**Reasons**
1) $p \to r$	Premise
2) $\neg r \to \neg p$	Step (1) and $p \to r \Longleftrightarrow \neg r \to \neg p$
3) $\neg p \to q$	Premise
4) $\neg r \to q$	Steps (2) and (3) and the Law of the Syllogism
5) $q \to s$	Premise
6) $\therefore \neg r \to s$	Steps (4) and (5) and the Law of the Syllogism

A second way to validate the given argument proceeds as follows.

Steps	**Reasons**
1) $p \to r$	Premise
2) $q \to s$	Premise
3) $\neg p \to q$	Premise
4) $p \vee q$	Step (3) and $(\neg p \to q) \Longleftrightarrow (\neg\neg p \vee q) \Longleftrightarrow (p \vee q)$, where the second logical equivalence follows by the Law of Double Negation
5) $r \vee s$	Steps (1), (2), and (4) and the Rule of the Constructive Dilemma
6) $\therefore \neg r \to s$	Step (5) and $(r \vee s) \Longleftrightarrow (\neg\neg r \vee s) \Longleftrightarrow (\neg r \to s)$, where the Law of Double Negation is used in the first logical equivalence

---

The next example is somewhat more involved.

Table 2.19

Rule of Inference	Related Logical Implication	Name of Rule
1) p $p \rightarrow q$ $\therefore q$	$[p \wedge (p \rightarrow q)] \rightarrow q$	Rule of Detachment (Modus Ponens)
2) $p \rightarrow q$ $q \rightarrow r$ $\therefore p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$	Law of the Syllogism
3) $p \rightarrow q$ $\neg q$ $\therefore \neg p$	$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$	Modus Tollens
4) p q $\therefore p \wedge q$		Rule of Conjunction
5) $p \vee q$ $\neg p$ $\therefore q$	$[(p \vee q) \wedge \neg p] \rightarrow q$	Rule of Disjunctive Syllogism
6) $\neg p \rightarrow F_0$ $\therefore p$	$(\neg p \rightarrow F_0) \rightarrow p$	Rule of Contradiction
7) $p \wedge q$ $\therefore p$	$(p \wedge q) \rightarrow p$	Rule of Conjunctive Simplification
8) p $\therefore p \vee q$	$p \rightarrow p \vee q$	Rule of Disjunctive Amplification
9) $p \wedge q$ $p \rightarrow (q \rightarrow r)$ $\therefore r$	$[(p \wedge q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow r$	Rule of Conditional Proof
10) $p \rightarrow r$ $q \rightarrow r$ $\therefore (p \vee q) \rightarrow r$	$[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$	Rule for Proof by Cases
11) $p \rightarrow q$ $r \rightarrow s$ $p \vee r$ $\therefore q \vee s$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$	Rule of the Constructive Dilemma
12) $p \rightarrow q$ $r \rightarrow s$ $\neg q \vee \neg s$ $\therefore \neg p \vee \neg r$	$[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$	Rule of the Destructive Dilemma

EXAMPLE 2.30 Establish the validity of the argument

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow (r \wedge s) \\ \neg r \vee (\neg t \vee u) \\ p \wedge t \\ \hline \therefore u \end{array}$$

Steps	Reasons
1) $p \rightarrow q$	Premise
2) $q \rightarrow (r \wedge s)$	Premise
3) $p \rightarrow (r \wedge s)$	Steps (1) and (2) and the Law of the Syllogism
4) $p \wedge t$	Premise
5) p	Step (4) and the Rule of Conjunctive Simplification
6) $r \wedge s$	Steps (5) and (3) and the Rule of Detachment
7) r	Step (6) and the Rule of Conjunctive Simplification
8) $\neg r \vee (\neg t \vee u)$	Premise
9) $\neg(r \wedge t) \vee u$	Step (8), the Associative Law of $\vee$, and DeMorgan's Laws
10) t	Step (4) and the Rule of Conjunctive Simplification
11) $r \wedge t$	Steps (7) and (10) and the Rule of Conjunction
12) $\therefore u$	Steps (9) and (11), the Law of Double Negation, and the Rule of Disjunctive Syllogism

EXAMPLE 2.31

This example will provide a way to show that the following argument is valid.

> If the band could not play rock music or the refreshments were not delivered on time, then the New Year's party would have been canceled and Alicia would have been angry. If the party were canceled, then refunds would have had to be made. No refunds were made.
>
> Therefore the band could play rock music.

First we convert the given argument into symbolic form by using the following statement assignments:

p: The band could play rock music.
q: The refreshments were delivered on time.
r: The New Year's party was canceled.
s: Alicia was angry.
t: Refunds had to be made.

The argument above now becomes

$$\begin{array}{l} (\neg p \vee \neg q) \rightarrow (r \wedge s) \\ r \rightarrow t \\ \underline{\neg t \qquad\qquad\qquad} \\ \therefore p \end{array}$$

We can establish the validity of this argument as follows.

Steps	Reasons
1) $r \rightarrow t$	Premise
2) $\neg t$	Premise
3) $\neg r$	Steps (1) and (2) and Modus Tollens
4) $\neg r \vee \neg s$	Step (3) and the Rule of Disjunctive Amplification
5) $\neg(r \wedge s)$	Step (4) and DeMorgan's Laws
6) $(\neg p \vee \neg q) \rightarrow (r \wedge s)$	Premise
7) $\neg(\neg p \vee \neg q)$	Steps (6) and (5) and Modus Tollens
8) $p \wedge q$	Step (7), DeMorgan's Laws, and the Law of Double Negation
9) $\therefore p$	Step (8) and the Rule of Conjunctive Simplification

EXAMPLE 2.32

In this instance we shall use the method of Proof by Contradiction. Consider the argument

$$\begin{array}{l} \neg p \leftrightarrow q \\ q \rightarrow r \\ \underline{\neg r} \\ \therefore p \end{array}$$

To establish the validity for this argument, we assume the negation $\neg p$ of the conclusion p as another premise. The objective now is to use these four premises to derive a contradiction F_0. Our derivation follows.

Steps	**Reasons**
1) $\neg p \leftrightarrow q$	Premise
2) $(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)$	Step (1) and $(\neg p \leftrightarrow q) \Longleftrightarrow [(\neg p \rightarrow q) \wedge (q \rightarrow \neg p)]$
3) $\neg p \rightarrow q$	Step (2) and the Rule of Conjunctive Simplification
4) $q \rightarrow r$	Premise
5) $\neg p \rightarrow r$	Steps (3) and (4) and the Law of the Syllogism
6) $\neg p$	Premise (the one assumed)
7) r	Steps (5) and (6) and the Rule of Detachment
8) $\neg r$	Premise
9) $r \wedge \neg r (\Longleftrightarrow F_0)$	Steps (7) and (8) and the Rule of Conjunction
10) $\therefore p$	Steps (6) and (9) and the method of Proof by Contradiction

If we examine further what has happened here, we find that

$$[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p] \Rightarrow F_0.$$

This requires the truth value of $[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p]$ to be 0. Because $\neg p \leftrightarrow q$, $q \rightarrow r$, and $\neg r$ are the given premises, each of these statements has the truth value 1. Consequently, for $[(\neg p \leftrightarrow q) \wedge (q \rightarrow r) \wedge \neg r \wedge \neg p]$ to have the truth value 0, the statement $\neg p$ must have the truth value 0. Therefore p has the truth value 1, and the conclusion p of the argument is true.

Before we consider our next example, we need to examine columns 5 and 7 of Table 2.20. These identical columns tell us that for primitive statements p, q, and r,

$$[p \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p \wedge q) \rightarrow r].$$

Using the first substitution rule, let us replace each occurrence of p by the compound statement $(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$. Then we obtain the new result

$$[(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q)^{\dagger} \rightarrow r].$$

---

† In Section 4.2 we shall present a formal proof of why

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \wedge q \Longleftrightarrow p_1 \wedge p_2 \wedge \cdots \wedge p_n \wedge q.$$

Table 2.20

p	q	r	$p \wedge q$	$(p \wedge q) \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
0	0	0	0	1	1	1
0	0	1	0	1	1	1
0	1	0	0	1	0	1
0	1	1	0	1	1	1
1	0	0	0	1	1	1
1	0	1	0	1	1	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

This result tells us that if we wish to establish the validity of the argument (*) we may be able to do so by establishing the validity of the corresponding argument (**).

$$
(*)\quad \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \rightarrow r \end{array}
\qquad\qquad
(**)\quad \begin{array}{l} p_1 \\ p_2 \\ \vdots \\ p_n \\ q \\ \hline \therefore r \end{array}
$$

After all, suppose we want to show that $q \rightarrow r$ has the truth value 1, when each of $p_1, p_2, \ldots, p_n$ does. If the truth value for q is 0, then there is nothing left to do, since the truth value for $q \rightarrow r$ is 1. Hence the real problem is to show that $q \rightarrow r$ has truth value 1, when each of $p_1, p_2, \ldots, p_n$, *and* q does — that is, we need to show that when $p_1, p_2, \ldots, p_n, q$ each have truth value 1, then the truth value of r is 1.

We demonstrate this principle in the next example.

EXAMPLE 2.33

In order to establish the validity of the argument

$$
(*)\qquad \begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ \hline \therefore q \rightarrow p \end{array}
$$

we consider the corresponding argument

$$
(**)\qquad \begin{array}{l} u \rightarrow r \\ (r \wedge s) \rightarrow (p \vee t) \\ q \rightarrow (u \wedge s) \\ \neg t \\ q \\ \hline \therefore p \end{array}
$$

[Note that q is the hypothesis of the conclusion $q \rightarrow p$ for argument (*) and that it becomes another premise for argument (**) where the conclusion is p.]

To validate the argument (**) we proceed as follows.

Steps	Reasons
1) q	Premise
2) $q \rightarrow (u \wedge s)$	Premise
3) $u \wedge s$	Steps (1) and (2) and the Rule of Detachment
4) u	Step (3) and the Rule of Conjunctive Simplification
5) $u \rightarrow r$	Premise
6) r	Steps (4) and (5) and the Rule of Detachment
7) s	Step (3) and the Rule of Conjunctive Simplification
8) $r \wedge s$	Steps (6) and (7) and the Rule of Conjunction
9) $(r \wedge s) \rightarrow (p \vee t)$	Premise
10) $p \vee t$	Steps (8) and (9) and the Rule of Detachment
11) $\neg t$	Premise
12) $\therefore p$	Steps (10) and (11) and the Rule of Disjunctive Syllogism

We now know that for argument (**)

$$[(u \rightarrow r) \wedge [(r \wedge s) \rightarrow (p \vee t)] \wedge [q \rightarrow (u \wedge s)] \wedge \neg t \wedge q] \Rightarrow p,$$

and for argument (*) it follows that

$$[(u \rightarrow r) \wedge [(r \wedge s) \rightarrow (p \vee t)] \wedge [q \rightarrow (u \wedge s)] \wedge \neg t] \Rightarrow (q \rightarrow p).$$

Examples 2.29 through 2.33 have given us some idea of how to establish the validity of an argument. Following Example 2.25 we discussed two situations indicating when an argument is invalid — namely, when we try to argue by the converse or the inverse. So now it is time for us to learn a little more about how to determine when an argument is invalid.

Given an argument

$$\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ \underline{p_n} \\ \therefore q \end{array}$$

we say that the argument is invalid if it is possible for each of the premises $p_1, p_2, p_3, \ldots, p_n$ to be true (with truth value 1), while the conclusion q is false (with truth value 0).

The next example illustrates an indirect method whereby we may be able to show that an argument we *feel* is invalid (perhaps because we cannot find a way to show that it is valid) actually *is* invalid.

EXAMPLE 2.34

Consider the primitive statements p, q, r, s, and t and the argument

$$\begin{array}{l} p \\ p \vee q \\ q \rightarrow (r \rightarrow s) \\ \underline{t \rightarrow r} \\ \therefore \neg s \rightarrow \neg t \end{array}$$

To show that this is an invalid argument, we need *one* assignment of truth values for each of the statements p, q, r, s, and t such that the conclusion $\neg s \rightarrow \neg t$ is false (has the truth value 0) while the four premises are all true (have the truth value 1). The only time the conclusion $\neg s \rightarrow \neg t$ is false is when $\neg s$ is true and $\neg t$ is false. This implies that the truth value for s is 0 and that the truth value for t is 1.

Because p is one of the premises, its truth value must be 1. For the premise $p \vee q$ to have the truth value 1, q may be either true (1) or false (0). So let us consider the premise $t \rightarrow r$ where we know that t is true. If $t \rightarrow r$ is to be true, then r must be true (have the truth value 1). Now with r true (1) and s false (0), it follows that $r \rightarrow s$ is false (0), and that the truth value of the premise $q \rightarrow (r \rightarrow s)$ will be 1 only when q is false (0).

Consequently, under the truth value assignments

$$p:\ 1 \qquad q:\ 0 \qquad r:\ 1 \qquad s:\ 0 \qquad t:\ 1,$$

the four premises

$$p \qquad p \vee q \qquad q \rightarrow (r \rightarrow s) \qquad t \rightarrow r$$

all have the truth value 1, while the conclusion

$$\neg s \rightarrow \neg t$$

has the truth value 0. In this case we have shown the given argument to be invalid.

The truth value assignments p: 1, q: 0, r: 1, s: 0, and t: 1 of Example 2.34 provide one case that *disproves* what we thought might have been a valid argument. We should now start to realize that in trying to show that an implication of the form

$$(p_1 \wedge p_2 \wedge p_3 \wedge \cdots \wedge p_n) \rightarrow q$$

presents a valid argument, we need to consider *all* cases where the premises $p_1, p_2, p_3, \ldots, p_n$ are true. [Each such case is an assignment of truth values for the primitive statements (that make up the premises) where $p_1, p_2, p_3, \ldots, p_n$ are true.] In order to do so — namely, to cover the cases without writing out the truth table — we have been using the rules of inference together with the laws of logic and other logical equivalences. To cover all the necessary cases, we cannot use one specific example (or case) as a means of establishing the validity of the argument (for all possible cases). However, whenever we wish to show that an implication (of the preceding form) is not a tautology, all we need to find is one case for which the implication is false — that is, one case in which all the premises are true but the conclusion is false. This *one* case provides a *counterexample* for the argument and shows it to be invalid.

Let us consider a second example wherein we try the indirect approach of Example 2.34.

EXAMPLE 2.35

What can we say about the validity or invalidity of the following argument? Here p, q, r, and s denote primitive statements.)

$$\begin{array}{l} p \rightarrow q \\ q \rightarrow s \\ r \rightarrow \neg s \\ \underline{\neg p \veebar r} \\ \therefore \neg p \end{array}$$

Can the conclusion $\neg p$ be false while the four premises are all true? The conclusion $\neg p$ is false when p has the truth value 1. So for the premise $p \rightarrow q$ to be true, the truth value of q must be 1. From the truth of the premise $q \rightarrow s$, the truth of q forces the truth of s. Consequently, at this point we have statements p, q, and s all with the truth value 1.

Continuing with the premise $r \rightarrow \neg s$, we find that because s has the truth value 1, the truth value of r must be 0. Hence r is false. But with $\neg p$ false and the premise $\neg p \veebar r$ true, we also have r true. Therefore we find that $p \Rightarrow (\neg r \wedge r)$.

We have failed in our attempt to find a counterexample to the validity of the given argument. However, this failure has shown us that the given argument is valid — and the validity follows by using the method of Proof by Contradiction.

This introduction to the rules of inference has been far from exhaustive. Several of the books cited among the references listed near the end of this chapter offer additional material for the reader who wishes to pursue this topic further. In Section 2.5 we shall apply the ideas developed in this section to statements of a more mathematical nature. For we shall want to learn how to develop a proof for a theorem. And then in Chapter 4 another very important proof technique called *mathematical induction* will be added to our arsenal of weapons for proving mathematical theorems. First, however, the reader should carefully complete the exercises for this section.

EXERCISES 2.3

1. The following are three valid arguments. Establish the validity of each by means of a truth table. In each case, determine which rows of the table are crucial for assessing the validity of the argument and which rows can be ignored.

a) $[p \wedge (p \rightarrow q) \wedge r] \rightarrow [(p \vee q) \rightarrow r]$

b) $[[(p \wedge q) \rightarrow r] \wedge \neg q \wedge (p \rightarrow \neg r)] \rightarrow (\neg p \vee \neg q)$

c) $[[p \vee (q \vee r)] \wedge \neg q] \rightarrow (p \vee r)$

2. Use truth tables to verify that each of the following is a logical implication.

a) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

b) $[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$

c) $[(p \vee q) \wedge \neg p] \rightarrow q$

d) $[(p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow [(p \vee q) \rightarrow r]$

3. Verify that each of the following is a logical implication by showing that it is impossible for the conclusion to have the truth value 0 while the hypothesis has the truth value 1.

a) $(p \wedge q) \rightarrow p$

b) $p \rightarrow (p \vee q)$

c) $[(p \vee q) \wedge \neg p] \rightarrow q$

d) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$

e) $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (\neg q \vee \neg s)] \rightarrow (\neg p \vee \neg r)$

4. For each of the following pairs of statements, use Modus Ponens or Modus Tollens to fill in the blank line so that a valid argument is presented.

a) If Janice has trouble starting her car, then her daughter Angela will check Janice's spark plugs.
Janice had trouble starting her car.
∴ ______________________

b) If Brady solved the first problem correctly, then the answer he obtained is 137.
Brady's answer to the first problem is not 137.
∴ ______________________

c) If this is a **repeat-until** loop, then the body of this loop is executed at least once.
____
∴ The body of the loop is executed at least once.

d) If Tim plays basketball in the afternoon, then he will not watch television in the evening.
____
∴ Tim didn't play basketball in the afternoon.

5. Consider each of the following arguments. If the argument is valid, identify the rule of inference that establishes its validity. If not, indicate whether the error is due to an attempt to argue by the converse or by the inverse.

a) Andrea can program in C++, and she can program in Java.
Therefore Andrea can program in C++.

b) A sufficient condition for Bubbles to win the golf tournament is that her opponent Meg not sink a birdie on the last hole.
Bubbles won the golf tournament.
Therefore Bubbles' opponent Meg did not sink a birdie on the last hole.

c) If Ron's computer program is correct, then he'll be able to complete his computer science assignment in at most two hours.
It takes Ron over two hours to complete his computer science assignment.
Therefore Ron's computer program is not correct.

d) Eileen's car keys are in her purse, or they are on the kitchen table.

Eileen's car keys are not on the kitchen table.
Therefore Eileen's car keys are in her purse.

e) If interest rates fall, then the stock market will rise.
Interest rates are not falling.
Therefore the stock market will not rise.

6. For primitive statements p, q, and r, let P denote the statement

$$[p \wedge (q \wedge r)] \vee \neg[p \vee (q \wedge r)],$$

while P_1 denotes the statement

$$[p \wedge (q \vee r)] \vee \neg[p \vee (q \vee r)].$$

a) Use the rules of inference to show that

$$q \wedge r \Rightarrow q \vee r.$$

b) Is it true that $P \Rightarrow P_1$?

7. Give the reason(s) for each step needed to show that the following argument is valid.

$$[p \wedge (p \rightarrow q) \wedge (s \vee r) \wedge (r \rightarrow \neg q)] \rightarrow (s \vee t)$$

Steps	Reasons
1) p	
2) $p \rightarrow q$	
3) q	
4) $r \rightarrow \neg q$	
5) $q \rightarrow \neg r$	
6) $\neg r$	
7) $s \vee r$	
8) s	
9) $\therefore s \vee t$	

8. Give the reasons for the steps verifying the following argument.

$$\begin{array}{l} (\neg p \vee q) \rightarrow r \\ r \rightarrow (s \vee t) \\ \neg s \wedge \neg u \\ \neg u \rightarrow \neg t \\ \hline \therefore p \end{array}$$

Steps	Reasons
1) $\neg s \wedge \neg u$	
2) $\neg u$	
3) $\neg u \rightarrow \neg t$	
4) $\neg t$	
5) $\neg s$	
6) $\neg s \wedge \neg t$	
7) $r \rightarrow (s \vee t)$	
8) $\neg(s \vee t) \rightarrow \neg r$	
9) $(\neg s \wedge \neg t) \rightarrow \neg r$	
10) $\neg r$	
11) $(\neg p \vee q) \rightarrow r$	
12) $\neg r \rightarrow \neg(\neg p \vee q)$	
13) $\neg r \rightarrow (p \wedge \neg q)$	
14) $p \wedge \neg q$	
15) $\therefore p$	

9. a) Give the reasons for the steps given to validate the argument

$$[(p \rightarrow q) \wedge (\neg r \vee s) \wedge (p \vee r)] \rightarrow (\neg q \rightarrow s).$$

Steps	Reasons
1) $\neg(\neg q \rightarrow s)$	
2) $\neg q \wedge \neg s$	
3) $\neg s$	
4) $\neg r \vee s$	
5) $\neg r$	
6) $p \rightarrow q$	
7) $\neg q$	
8) $\neg p$	
9) $p \vee r$	
10) r	
11) $\neg r \wedge r$	
12) $\therefore \neg q \rightarrow s$	

b) Give a direct proof for the result in part (a).

c) Give a direct proof for the result in Example 2.32.

10. Establish the validity of the following arguments.

a) $[(p \wedge \neg q) \wedge r] \rightarrow [(p \wedge r) \vee q]$

b) $[p \wedge (p \rightarrow q) \wedge (\neg q \vee r)] \rightarrow r$

c) $\begin{array}{l} p \rightarrow q \\ \neg q \\ \neg r \\ \hline \therefore \neg(p \vee r) \end{array}$

d) $\begin{array}{l} p \rightarrow q \\ r \rightarrow \neg q \\ r \\ \hline \therefore \neg p \end{array}$

e) $\begin{array}{l} p \rightarrow (q \rightarrow r) \\ \neg q \rightarrow \neg p \\ p \\ \hline \therefore r \end{array}$

f) $\begin{array}{l} p \wedge q \\ p \rightarrow (r \wedge q) \\ r \rightarrow (s \vee t) \\ \neg s \\ \hline \therefore t \end{array}$

g) $\begin{array}{l} p \rightarrow (q \rightarrow r) \\ p \vee s \\ t \rightarrow q \\ \neg s \\ \hline \therefore \neg r \rightarrow \neg t \end{array}$

h) $\begin{array}{l} p \vee q \\ \neg p \vee r \\ \neg r \\ \hline \therefore q \end{array}$

11. Show that each of the following arguments is invalid by providing a counterexample—that is, an assignment of truth values for the given primitive statements p, q, r, and s such that all premises are true (have the truth value 1) while the conclusion is false (has the truth value 0).

a) $[(p \wedge \neg q) \wedge [p \rightarrow (q \rightarrow r)]] \rightarrow \neg r$

b) $[[(p \wedge q) \rightarrow r] \wedge (\neg q \vee r)] \rightarrow p$

c) $\begin{array}{l} p \leftrightarrow q \\ q \rightarrow r \\ r \vee \neg s \\ \neg s \rightarrow q \\ \hline \therefore s \end{array}$

d) $\begin{array}{l} p \\ p \rightarrow r \\ p \rightarrow (q \vee \neg r) \\ \neg q \vee \neg s \\ \hline \therefore s \end{array}$

12. Write each of the following arguments in symbolic form. Then establish the validity of the argument or give a counterexample to show that it is invalid.

a) If Rochelle gets the supervisor's position and works hard, then she'll get a raise. If she gets the raise, then she'll buy a new car. She has not purchased a new car. Therefore either Rochelle did not get the supervisor's position or she did not work hard.

b) If Dominic goes to the racetrack, then Helen will be mad. If Ralph plays cards all night, then Carmela will be mad. If either Helen or Carmela gets mad, then Veronica (their attorney) will be notified. Veronica has not heard from either of these two clients. Consequently, Dominic didn't make it to the racetrack and Ralph didn't play cards all night.

c) If there is a chance of rain or her red headband is missing, then Lois will not mow her lawn. Whenever the temperature is over 80°F, there is no chance for rain. Today the temperature is 85°F and Lois is wearing her red headband. Therefore (sometime today) Lois will mow her lawn.

13. a) Given primitive statements p, q, r, show that the implication

$$[(p \vee q) \wedge (\neg p \vee r)] \rightarrow (q \vee r)$$

is a tautology.

b) The tautology in part (a) provides the rule of inference known as *resolution*, where the conclusion $(q \vee r)$ is called the *resolvent*. This rule was proposed in 1965 by J. A. Robinson and is the basis of many computer programs designed to automate a reasoning system.

In applying resolution each premise (in the hypothesis) and the conclusion are written as *clauses*. A clause is a primitive statement or its negation, or it is the disjunction of terms each of which is a primitive statement or the negation of such a statement. Hence the given rule has the clauses $(p \vee q)$ and $(\neg p \vee r)$ as premises and the clause $(q \vee r)$ as its conclusion (or, resolvent). Should we have the premise $\neg(p \wedge q)$, we replace this by the logically equivalent clause $\neg p \vee \neg q$, by the first of DeMorgan's Laws. The premise $\neg(p \vee q)$ can be replaced by the two clauses $\neg p$, $\neg q$. This is due to the second DeMorgan Law and the Rule of Conjunctive Simplification. For the premise $p \vee (q \wedge r)$, we apply the Distributive Law of $\vee$ over $\wedge$ and the Rule of Conjunctive Simplification to arrive at either of the two clauses $p \vee q$, $p \vee r$. Finally, the premise $p \rightarrow q$ becomes the clause $\neg p \vee q$.

Establish the validity of the following arguments, using resolution (along with the rules of inference and the laws of logic).

(i)
$$\begin{array}{l} p \vee (q \wedge r) \\ \underline{p \rightarrow s} \\ \therefore r \vee s \end{array}$$

(ii)
$$\begin{array}{l} p \\ \underline{p \leftrightarrow q} \\ \therefore q \end{array}$$

(iii)
$$\begin{array}{l} p \vee q \\ p \rightarrow r \\ \underline{r \rightarrow s} \\ \therefore q \vee s \end{array}$$

(iv)
$$\begin{array}{l} \neg p \vee q \vee r \\ \neg q \\ \underline{\neg r} \\ \therefore \neg p \end{array}$$

(v)
$$\begin{array}{l} \neg p \vee s \\ \neg t \vee (s \wedge r) \\ \neg q \vee r \\ \underline{p \vee q \vee t} \\ \therefore r \vee s \end{array}$$

c) Write the following argument in symbolic form, then use resolution (along with the rules of inference and the laws of logic) to establish its validity.

Jonathan does not have his driver's license or his new car is out of gas. Jonathan has his driver's license or he does not like to drive his new car. Jonathan's new car is not out of gas or he does not like to drive his new car. Therefore, Jonathan does not like to drive his new car.

2.4 The Use of Quantifiers

In Section 2.1, we mentioned how sentences that involve a variable, such as x, need not be statements. For example, the sentence "The number $x + 2$ is an even integer" is not necessarily true or false unless we know what value is substituted for x. If we restrict our choices to integers, then when x is replaced by -5, -1, or 3, for instance, the resulting statement is false. In fact, it is false whenever x is replaced by an odd integer. When an even integer is substituted for x, however, the resulting statement is true.

We refer to the sentence "The number $x + 2$ is an even integer" as an *open statement*, which we formally define as follows.

Definition 2.5 A declarative sentence is an *open statement* if

1) it contains one or more variables, and

2) it is not a statement, but

3) it becomes a statement when the variables in it are replaced by certain allowable choices.

When we examine the sentence "The number $x + 2$ is an even integer" in light of this definition, we find it is an open statement that contains the single variable x. With regard to the third element of the definition, in our earlier discussion we restricted the "certain allowable choices" to integers. These allowable choices constitute what is called the *universe* or *universe of discourse* for the open statement. The universe comprises the choices we wish to consider or allow for the variable(s) in the open statement. (The universe is an example of a *set*, a concept we shall examine in some detail in the next chapter.)

In dealing with open statements, we use the following notation:

The open statement "The number $x + 2$ is an even integer" is denoted by $p(x)$ [or $q(x)$, $r(x)$, etc.]. Then $\neg p(x)$ may be read "The number $x + 2$ is *not* an even integer."

We shall use $q(x, y)$ to represent an open statement that contains two variables. For example, consider

$q(x, y)$: The numbers $y + 2$, $x - y$, and $x + 2y$ are even integers.

In the case of $q(x, y)$, there is more than one occurrence of each of the variables x, y. It is understood that when we replace one of the x's by a choice from our universe, we replace the other x by the same choice. Likewise, when a substitution (from the universe) is made for one occurrence of y, that same substitution is made for all other occurrences of the variable y.

With $p(x)$ and $q(x, y)$ as above, and the universe still stipulating the integers as our only allowable choices, we get the following results when we make some replacements for the variables x, y.

$p(5)$: The number $7(= 5 + 2)$ is an even integer. (FALSE)

$\neg p(7)$: The number 9 is not an even integer. (TRUE)

$q(4, 2)$: The numbers 4, 2, and 8 are even integers. (TRUE)

We also note, for example, that $q(5, 2)$ and $q(4, 7)$ are both false statements, whereas $\neg q(5, 2)$ and $\neg q(4, 7)$ are true.

Consequently, we see that for both $p(x)$ and $q(x, y)$, as already given, some substitutions result in true statements and others in false statements. Therefore we can make the following true statements.

1) For some x, $p(x)$.

2) For some x, y, $q(x, y)$.

Note that in this situation, the statements "For some x, $\neg p(x)$" and "For some x, y, $\neg q(x, y)$" are also true. [Since the statements "For some x, $p(x)$" and "For some x, $\neg p(x)$" are both true, we realize that the second statement is *not* the negation of the first—even though the open statement $\neg p(x)$ is the negation of the open statement $p(x)$. And a similar result is true for the statements involving $q(x, y)$ and $\neg q(x, y)$.]

The phrases "For some x" and "For some x, y" are said to *quantify* the open statements $p(x)$ and $q(x, y)$, respectively. Many postulates, definitions, and theorems in mathematics involve statements that are quantified open statements. These result from the two types of *quantifiers*, which are called the *existential* and the *universal quantifiers*.

Statement (1) uses the *existential quantifier* "For some x," which can also be expressed as "For at least one x" or "There exists an x such that." This quantifier is written in symbolic form as $\exists x$. Hence the statement "For some x, $p(x)$" becomes $\exists x\ p(x)$, in symbolic form.

Statement (2) becomes $\exists x\ \exists y\ q(x, y)$ in symbolic form. The notation $\exists x, y$ can be used to abbreviate $\exists x\ \exists y\ q(x, y)$ to $\exists x, y\ q(x, y)$.

The *universal quantifier* is denoted by $\forall x$ and is read "For all x," "For any x," "For each x," or "For every x." "For all x, y," "For any x, y," "For every x, y," or "For all x and y" is denoted by $\forall x\ \forall y$, which can be abbreviated to $\forall x, y$.

Taking $p(x)$ as defined earlier and using the universal quantifier, we can change the open statement $p(x)$ into the (quantified) statement $\forall x\ p(x)$, a false statement.

If we consider the open statement $r(x)$: "$2x$ is an even integer" with the same universe (of all integers), then the (quantified) statement $\forall x\ r(x)$ is a true statement. When we say that $\forall x\ r(x)$ is true, we mean that no matter which integer (from our universe) is substituted for x in $r(x)$, the resulting statement is true. Also note that the statement $\exists x\ r(x)$ is a true statement, whereas $\forall x\ \neg r(x)$ and $\exists x\ \neg r(x)$ are both false.

The variable x in each of open statements $p(x)$ and $r(x)$ is called a *free variable* (of the open statement). As x varies over the universe for an open statement, the truth value of the statement (that results upon the replacement of each occurrence of x) may vary. For instance, in the case of $p(x)$, we found $p(5)$ to be false — while $p(6)$ turns out to be a true statement. The open statement $r(x)$, however, becomes a true statement for every replacement (for x) taken from the universe of all integers. In contrast to the open statement $p(x)$ the statement $\exists x\ p(x)$ has a fixed truth value — namely, true. And in the symbolic representation $\exists x\ p(x)$ the variable x is said to be a *bound* variable — it is bound by the existential quantifier $\exists$. This is also the case for the statements $\forall x\ r(x)$ and $\forall x\ \neg r(x)$, where in each case the variable x is bound by the universal quantifier $\forall$.

For the open statement $q(x, y)$ we have two free variables, each of which is bound by the quantifier $\exists$ in either of the statements $\exists x\ \exists y\ q(x, y)$ or $\exists x, y\ q(x, y)$.

The following example shows how these new ideas about quantifiers can be used in conjunction with the logical connectives.

EXAMPLE 2.36

Here the universe comprises all real numbers. The open statements $p(x)$, $q(x)$, $r(x)$, and $s(x)$ are given by

$$p(x):\ x \geq 0 \qquad r(x):\ x^2 - 3x - 4 = 0$$
$$q(x):\ x^2 \geq 0 \qquad s(x):\ x^2 - 3 > 0.$$

Then the following statements are true.

1) $$\exists x\ [p(x) \wedge r(x)]$$

This follows because the real number 4, for example, is a member of the universe and is such that both of the statements $p(4)$ and $r(4)$ are true.

2) $$\forall x\ [p(x) \rightarrow q(x)]$$

If we replace x in $p(x)$ by a negative real number a, then $p(a)$ is false, but $p(a) \rightarrow q(a)$ is true regardless of the truth value of $q(a)$. Replacing x in $p(x)$ by a nonnegative real number b, we find that $p(b)$ and $q(b)$ are both true, as is $p(b) \rightarrow q(b)$. Consequently, $p(x) \rightarrow q(x)$ is true for all replacements x taken from the universe of all real numbers, and the (quantified) statement $\forall x\ [p(x) \rightarrow q(x)]$ is true.

This statement may be translated into any of the following:

a) For every real number x, if $x \geq 0$, then $x^2 \geq 0$.

b) Every nonnegative real number has a nonnegative square.

c) The square of any nonnegative real number is a nonnegative real number.

d) All nonnegative real numbers have nonnegative squares.

Also, the statement $\exists x\ [p(x) \rightarrow q(x)]$ is true.

The next statements we examine are false.

1′) $$\forall x\ [q(x) \rightarrow s(x)]$$

We want to show that the statement is false, so we need exhibit only one *counterexample* — that is, *one value of* x for which $q(x) \rightarrow s(x)$ is false — rather than prove something for all x as we did for statement (2). Replacing x by 1, we find that $q(1)$ is true and $s(1)$ is false. Therefore $q(1) \rightarrow s(1)$ is false, and consequently the (quantified) statement $\forall x\ [q(x) \rightarrow s(x)]$ is false. [Note that $x = 1$ does not produce the only counterexample: Every real number a between $-\sqrt{3}$ and $\sqrt{3}$ will make $q(a)$ true and $s(a)$ false.]

2′) $$\forall x\ [r(x) \vee s(x)]$$

Here there are many values for x, such as $1, \frac{1}{2}, -\frac{3}{2}$, and 0, that produce counterexamples. Upon changing quantifiers, however, we find that the statement $\exists x\ [r(x) \vee s(x)]$ is true.

3′) $$\forall x\ [r(x) \rightarrow p(x)]$$

The real number -1 is a solution of the equation $x^2 - 3x - 4 = 0$, so $r(-1)$ is true while $p(-1)$ is false. Therefore the choice of -1 provides the unique counterexample we need to show that this (quantified) statement is false.

Statement (3′) may be translated into either of the following:

a) For every real number x, if $x^2 - 3x - 4 = 0$, then $x \geq 0$.

b) For every real number x, if x is a solution of the equation $x^2 - 3x - 4 = 0$, then $x \geq 0$.

Now we make the following observations. Let $p(x)$ denote any open statement (in the variable x) with a prescribed *nonempty* universe (that is, the universe contains at least one member). Then if $\forall x\ p(x)$ is true, so is $\exists x\ p(x)$, or

$$\forall x\ p(x) \Rightarrow \exists x\ p(x).$$

When we write $\forall x\ p(x) \Rightarrow \exists x\ p(x)$ we are saying that the implication $\forall x\ p(x) \rightarrow \exists x\ p(x)$ is a logical implication — that is, $\exists x\ p(x)$ is true whenever $\forall x\ p(x)$ is true. Also, we realize that the hypothesis of this implication is the quantified *statement* $\forall x\ p(x)$, and the conclusion is $\exists x\ p(x)$, another quantified *statement*. On the other hand, it does not follow that if $\exists x\ p(x)$ is true, then $\forall x\ p(x)$ must be true. Hence $\exists x\ p(x)$ does not logically imply $\forall x\ p(x)$, in general.

Our next example brings out the fact that the quantification of an open statement may not be as explicit as we might prefer.

EXAMPLE 2.37

a) Let us consider the universe of all real numbers and examine the sentences:

1) If a number is rational, then it is a real number.

2) If x is rational, then x is real.

We should agree that these sentences convey the same information. But we should also question whether the sentences are statements or open statements. In the case of sentence (2) we at least have the presence of the variable x. But neither sentence contains an expression such as "For all," or "For every," or "For each." Our one and only clue to indicate that we are dealing with universally quantified statements here is the presence of the indefinite article "a" in the first sentence. In situations like these the use of the universal quantifier is *implicit* as opposed to *explicit*.

If we let $p(x)$, $q(x)$ be the open statements

$p(x)$: x is a rational number $\qquad$ $q(x)$: x is a real number,

then we must recognize the fact that both of the given sentences are somewhat informal ways of expressing the quantified statement

$$\forall x\ [p(x) \rightarrow q(x)].$$

b) For the universe of all triangles in the plane, the sentence

"An equilateral triangle has three angles of 60°, and conversely."

provides another instance of implicit quantification. Here the indefinite article "An" is the only indication that we might be able to express this sentence as a statement with a universal quantifier. If the open statements

$e(t)$: Triangle t is equilateral.

$a(t)$: Triangle t has three angles of 60°.

are defined for this universe, then the given sentence can be written in the explicit quantified form

$$\forall t\ [e(t) \leftrightarrow a(t)].$$

c) In the typical trigonometry textbook one often comes across the trigonometric identity

$$\sin^2 x + \cos^2 x = 1.$$

This identify contains no explicit quantification, and the reader must understand or be told that it is defined for all real numbers x. When the universe of all real numbers is specified (or at least understood), then the identity can be expressed by the (explicitly) quantified statement

$$\forall x\ [\sin^2 x + \cos^2 x = 1].$$

d) Finally, consider the universe of all positive integers and the sentence

"The integer 41 is equal to the sum of two perfect squares."

Here we have one more example where the quantification is implicit — but this time the quantification is existential. We may express the result here in a more formal (and symbolic) manner as

$$\exists m\ \exists n\ [41 = m^2 + n^2].$$

The next example demonstrates that the truth value of a quantified statement may depend on the universe prescribed.

EXAMPLE 2.38

Consider the open statement $p(x)$: $x^2 \geq 1$.

1) If the universe consists of all positive integers, then the quantified statement $\forall x\ p(x)$ is true.
2) For the universe of all positive real numbers, however, the same quantified statement $\forall x\ p(x)$ is false. The positive real number $1/2$ provides one of many possible counterexamples.

Yet for either universe, the quantified statement $\exists x\ p(x)$ is true.

One use of quantifiers in a computer science setting is illustrated in the following example.

EXAMPLE 2.39

In the following program segment, n is an integer variable and the variable A is an array $A[1], A[2], \ldots, A[20]$ of 20 integer values.

```
for n := 1 to 20 do
  A[n] := n * n - n
```

The following statements about the array A can be represented in quantified form, where the universe consists of all integers from 1 to 20, inclusive.

1) Every entry in the array is nonnegative:
$$\forall n\ (A[n] \geq 0).$$
2) There exist two consecutive entries in A where the larger entry is twice the smaller:
$$\exists n\ (A[n+1] = 2A[n]).$$
3) The entries in the array are sorted in (strictly) ascending order:
$$\forall n\ [(1 \leq n \leq 19) \rightarrow (A[n] < A[n+1])].$$
Our last statement requires the use of two integer variables m, n.
4) The entries in the array are distinct:
$$\forall m\ \forall n\ [(m \neq n) \rightarrow (A[m] \neq A[n])], \quad \text{or}$$
$$\forall m, n\ [(m < n) \rightarrow (A[m] \neq A[n])].$$

Before continuing, we summarize and somewhat extend, in Table 2.21, what we have learned about quantifiers.

The results in Table 2.21 may appear to involve only one open statement. However, we should realize that the open statement $p(x)$ in the table may stand for a conjunction of open statements, such as $q(x) \wedge r(x)$, or an implication of open statements, such as $s(x) \rightarrow t(x)$. If, for example, we want to know when the statement $\exists x\ [s(x) \rightarrow t(x)]$ is true, then we look at the table for $\exists x\ p(x)$ and use the information provided there. The table tells us that $\exists x\ [s(x) \rightarrow t(x)]$ is true when $s(a) \rightarrow t(a)$ is true for some (at least one) a in the prescribed universe.

We will look further into quantified statements involving more than one open statement. Before doing so, however, we need to examine the following definition. This definition is comparable to Definitions 2.2 and 2.4 where we defined the ideas of logically equivalent statements and logical implication. It settles the same types of questions for open statements.

Table 2.21

Statement	When Is It True?	When Is It False?
$\exists x\ p(x)$	For some (at least one) a in the universe, $p(a)$ is true.	For every a in the universe, $p(a)$ is false.
$\forall x\ p(x)$	For every replacement a from the universe, $p(a)$ is true.	There is at least one replacement a from the universe for which $p(a)$ is false.
$\exists x\ \neg p(x)$	For at least one choice a in the universe, $p(a)$ is false, so its negation $\neg p(a)$ is true.	For every replacement a in the universe, $p(a)$ is true.
$\forall x\ \neg p(x)$	For every replacement a from the universe, $p(a)$ is false and its negation $\neg p(a)$ is true.	There is at least one replacement a from the universe for which $\neg p(a)$ is false and $p(a)$ is true.

Definition 2.6 Let $p(x)$, $q(x)$ be open statements defined for a given universe.

The open statements $p(x)$ and $q(x)$ are called (*logically*) *equivalent*, and we write $\forall x\ [p(x) \Longleftrightarrow q(x)]$ when the biconditional $p(a) \leftrightarrow q(a)$ is true for each replacement a from the universe (that is, $p(a) \Longleftrightarrow q(a)$ for each a in the universe). If the implication $p(a) \rightarrow q(a)$ is true for each a in the universe (that is, $p(a) \Rightarrow q(a)$ for each a in the universe), then we write $\forall x\ [p(x) \Rightarrow q(x)]$ and say that $p(x)$ *logically implies* $q(x)$.

For the universe of all triangles in the plane, let $p(x)$, $q(x)$ denote the open statements

$$p(x):\quad x \text{ is equiangular} \qquad q(x):\quad x \text{ is equilateral.}$$

Then for every particular triangle a (a replacement for x) we know that $p(a) \leftrightarrow q(a)$ is true (that is, $p(a) \Longleftrightarrow q(a)$, for every triangle in the plane). Consequently, $\forall x\ [p(x) \Longleftrightarrow q(x)]$.

Observe that here and, in general, $\forall x\ [p(x) \Longleftrightarrow q(x)]$ if and only if $\forall x\ [p(x) \Rightarrow q(x)]$ and $\forall x\ [q(x) \Rightarrow p(x)]$.

We also realize that a definition similar to Definition 2.6 can be given for two open statements that involve two or more variables.

Now we take another look at the logical equivalence of statements (not open statements) as we examine the converse, inverse, and contrapositive of a statement of the form $\forall x\ [p(x) \rightarrow q(x)]$.

Definition 2.7 For open statements $p(x)$, $q(x)$ — defined for a prescribed universe — and the universally quantified statement $\forall x\ [p(x) \rightarrow q(x)]$, we define:

1) The *contrapositive* of $\forall x\ [p(x) \rightarrow q(x)]$ to be $\forall x\ [\neg q(x) \rightarrow \neg p(x)]$.

2) The *converse* of $\forall x\ [p(x) \rightarrow q(x)]$ to be $\forall x\ [q(x) \rightarrow p(x)]$.

3) The *inverse* of $\forall x\ [p(x) \rightarrow q(x)]$ to be $\forall x\ [\neg p(x) \rightarrow \neg q(x)]$.

The following two examples illustrate Definition 2.7.

EXAMPLE 2.40

For the universe of all quadrilaterals in the plane let $s(x)$ and $e(x)$ denote the open statements

$$s(x):\quad x \text{ is a square} \qquad e(x):\quad x \text{ is equilateral.}$$

a) The statement

$$\forall x\,[s(x) \to e(x)]$$

is a true statement and is logically equivalent to its contrapositive

$$\forall x\,[\neg e(x) \to \neg s(x)]$$

because $[s(a) \to e(a)] \Longleftrightarrow [\neg e(a) \to \neg s(a)]$ for each replacement a. Hence

$$\forall x\,[s(x) \to e(x)] \Longleftrightarrow \forall x\,[\neg e(x) \to \neg s(x)].$$

b) The statement

$$\forall x\,[e(x) \to s(x)]$$

is a false statement and is the converse of the true statement

$$\forall x\,[s(x) \to e(x)].$$

The false statement

$$\forall x\,[\neg s(x) \to \neg e(x)]$$

is the inverse of the given statement $\forall x\,[s(x) \to e(x)]$.

Since $[e(a) \to s(a)] \Longleftrightarrow [\neg s(a) \to \neg e(a)]$ for each specific quadrilateral a, we find that the converse and inverse are logically equivalent — that is,

$$\forall x\,[e(x) \to s(x)] \Longleftrightarrow \forall x\,[\neg s(x) \to \neg e(x)].$$

EXAMPLE 2.41

Here $p(x)$ and $q(x)$ are the open statements

$$p(x):\quad |x| > 3 \qquad q(x):\quad x > 3$$

and the universe consists of all real numbers.

a) The statement $\forall x\,[p(x) \to q(x)]$ is a false statement. For example, if $x = -5$, then $p(-5)$ is true while $q(-5)$ is false. Consequently, $p(-5) \to q(-5)$ is false, and so is $\forall x\,[p(x) \to q(x)]$.

b) We can express the converse of the given statement [in part (a)] as follows:

Every real number greater than 3 has magnitude
(or, absolute value) greater than 3.

In symbolic form this true statement is written $\forall x\,[q(x) \to p(x)]$.

c) The inverse of the given statement is also a true statement. In symbolic form we have $\forall x\,[\neg p(x) \to \neg q(x)]$, which can be expressed in words by

If the magnitude of a real number is less than or equal to 3,
then the number itself is less than or equal to 3.

And this is logically equivalent to the (converse) statement given in part (b).

d) Here the contrapositive of the statement in part (a) is given by $\forall x\,[\neg q(x) \to \neg p(x)]$. This false statement is logically equivalent to $\forall x\,[p(x) \to q(x)]$ and can be expressed as follows:

If a real number is less than or equal to 3, then so is its magnitude.

e) Together with $p(x)$ and $q(x)$ as above, consider the open statement

$$r(x): \quad x < -3,$$

which is also defined for the universe of all real numbers. The following four statements are all true:

Statement:	$\forall x\ [p(x) \rightarrow (r(x) \vee q(x))]$
Contrapositive:	$\forall x\ [\neg(r(x) \vee q(x)) \rightarrow \neg p(x)]$
Converse:	$\forall x\ [(r(x) \vee q(x)) \rightarrow p(x)]$
Inverse:	$\forall x\ [\neg p(x) \rightarrow \neg(r(x) \vee q(x))]$

In this case (because the statement and its converse are both true) we find that the statement $\forall x\ [p(x) \leftrightarrow (r(x) \vee q(x))]$ is true.

Now we use the results of Table 2.21 once again as we examine the next example.

EXAMPLE 2.42

Here the universe consists of all the integers, and the open statements $r(x)$, $s(x)$ are given by

$$r(x): \quad 2x + 1 = 5 \qquad s(x): \quad x^2 = 9.$$

We see that the statement $\exists x\ [r(x) \wedge s(x)]$ is false because there is no one integer a such that $2a + 1 = 5$ and $a^2 = 9$. However, there is an integer b $(= 2)$ such that $2b + 1 = 5$, and there is a second integer c $(= 3$ or $-3)$ such that $c^2 = 9$. Therefore the statement $\exists x\ r(x) \wedge \exists x\ s(x)$ is true. Consequently, the existential quantifier $\exists x$ does not distribute over the logical connective $\wedge$. This one counterexample is enough to show that

$$\exists x\ [r(x) \wedge s(x)] \not\Leftrightarrow [\exists x\ r(x) \wedge \exists x\ s(x)],$$

where $\not\Leftrightarrow$ is read "is *not* logically equivalent to." It also demonstrates that

$$[\exists x\ r(x) \wedge \exists x\ s(x)] \not\Rightarrow \exists x\ [r(x) \wedge s(x)],$$

where $\not\Rightarrow$ is read "does *not* logically imply." So the statement

$$[\exists x\ r(x) \wedge \exists x\ s(x)] \rightarrow \exists x\ [r(x) \wedge s(x)]$$

is *not* a tautology.

What, however, can we say about the converse of a quantified statement of this form? At this point we present a general argument for *any* (arbitrary) open statements $p(x)$, $q(x)$ and *any* (arbitrary) prescribed universe.

Examining the statement

$$\exists x\ [p(x) \wedge q(x)] \rightarrow [\exists x\ p(x) \wedge \exists x\ q(x)],$$

we find that when the hypothesis $\exists x\ [p(x) \wedge q(x)]$ is true, there is at least one element c in the universe for which the statement $p(c) \wedge q(c)$ is true. By the Rule of Conjunctive Simplification (see Section 2.3), $[p(c) \wedge q(c)] \Rightarrow p(c)$. From the truth of $p(c)$ we have the true statement $\exists x\ p(x)$. Similarly we obtain $\exists x\ q(x)$, another true statement. So $\exists x\ p(x) \wedge$

$\exists x\, q(x)$ is a true statement. Since $\exists x\, p(x) \wedge \exists x\, q(x)$ is true whenever $\exists x\, [p(x) \wedge q(x)]$ is true, it follows that

$$\exists x\, [p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)].$$

Arguments similar to the one for Example 2.42 provide the logical equivalences and logical implications listed in Table 2.22. In addition to those listed in Table 2.22 many other logical equivalences and logical implications can be derived.

Table 2.22 Logical Equivalences and Logical Implications for Quantified Statements in One Variable

For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x:
$\exists x\, [p(x) \wedge q(x)] \Rightarrow [\exists x\, p(x) \wedge \exists x\, q(x)]$
$\exists x\, [p(x) \vee q(x)] \Leftrightarrow [\exists x\, p(x) \vee \exists x\, q(x)]$
$\forall x\, [p(x) \wedge q(x)] \Leftrightarrow [\forall x\, p(x) \wedge \forall x\, q(x)]$
$[\forall x\, p(x) \vee \forall x\, q(x)] \Rightarrow \forall x\, [p(x) \vee q(x)]$

Our next example lists several of these and demonstrates how two of them are verified.

EXAMPLE 2.43

Let $p(x)$, $q(x)$, and $r(x)$ denote open statements for a given universe. We find the following logical equivalences. (Many more are also possible.)

1) $\forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)]$
To show that this statement is a logical equivalence we proceed as follows:
For each a in the universe, consider the *statements* $p(a) \wedge (q(a) \wedge r(a))$ and $(p(a) \wedge q(a)) \wedge r(a)$. By the Associative Law for $\wedge$, we have

$$p(a) \wedge (q(a) \wedge r(a)) \Leftrightarrow (p(a) \wedge q(a)) \wedge r(a).$$

Consequently, for the *open statements* $p(x) \wedge (q(x) \wedge r(x))$ and $(p(x) \wedge q(x)) \wedge r(x)$, it follows that

$$\forall x\, [p(x) \wedge (q(x) \wedge r(x))] \Leftrightarrow \forall x\, [(p(x) \wedge q(x)) \wedge r(x)].$$

2) $\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee q(x)]$
For each c in the universe, it follows from Example 2.7 that

$$[p(c) \rightarrow q(c)] \Leftrightarrow [\neg p(c) \vee q(c)].$$

Therefore the statement $\exists x\, [p(x) \rightarrow q(x)]$ is true (respectively, false) if and only if the statement $\exists x\, [\neg p(x) \vee q(x)]$ is true (respectively, false), so

$$\exists x\, [p(x) \rightarrow q(x)] \Leftrightarrow \exists x\, [\neg p(x) \vee q(x)].$$

3) Other logical equivalences that we shall often find useful include the following.
a) $\forall x\, \neg\neg p(x) \Leftrightarrow \forall x\, p(x)$
b) $\forall x\, \neg[p(x) \wedge q(x)] \Leftrightarrow \forall x\, [\neg p(x) \vee \neg q(x)]$
c) $\forall x\, \neg[p(x) \vee q(x)] \Leftrightarrow \forall x\, [\neg p(x) \wedge \neg q(x)]$

4) The results for the logical equivalences in 3(a), (b), and (c) remain valid when all of the universal quantifiers are replaced by existential quantifiers.

The results of Tables 2.21 and 2.22 and Examples 2.42 and 2.43 will now help us with a very important concept. How do we negate quantified statements that involve a single variable?

Consider the statement $\forall x\ p(x)$. Its negation—namely, $\neg[\forall x\ p(x)]$—can be stated as "It is not the case that for all x, $p(x)$ holds." This is not a very useful remark, so we consider $\neg[\forall x\ p(x)]$ further. When $\neg[\forall x\ p(x)]$ is true, then $\forall x\ p(x)$ is false, and so for some replacement a from the universe $\neg p(a)$ is true and $\exists x\ \neg p(x)$ is true. Conversely, whenever the statement $\exists x\ \neg p(x)$ is true we know that $\neg p(b)$ is true for some member b of the universe. Hence $\forall x\ p(x)$ is false and $\neg[\forall x\ p(x)]$ is true. So the statement $\neg[\forall x\ p(x)]$ is true if and only if the statement $\exists x\ \neg p(x)$ is true. (Similar considerations also tell us that $\neg[\forall x\ p(x)]$ is false if and only if $\exists x\ \neg p(x)$ is false.)

These observations lead to the following rule for negating the statement $\forall x\ p(x)$:

$$\neg[\forall x\ p(x)] \Longleftrightarrow \exists x\ \neg p(x).$$

In a similar way, Table 2.21 shows us that the statement $\exists x\ p(x)$ is true (false) precisely when the statement $\forall x\ \neg p(x)$ is false (true). This observation then motivates a rule for negating the statement $\exists x\ p(x)$:

$$\neg[\exists x\ p(x)] \Longleftrightarrow \forall x\ \neg p(x).$$

These two rules for negation, and two others that follow from them, are given in Table 2.23 for convenient reference.

Table 2.23 Rules for Negating Statements with One Quantifier

$\neg[\forall x\ p(x)] \Longleftrightarrow \exists x\ \neg p(x)$
$\neg[\exists x\ p(x)] \Longleftrightarrow \forall x\ \neg p(x)$
$\neg[\forall x\ \neg p(x)] \Longleftrightarrow \exists x\ \neg\neg p(x) \Longleftrightarrow \exists x\ p(x)$
$\neg[\exists x\ \neg p(x)] \Longleftrightarrow \forall x\ \neg\neg p(x) \Longleftrightarrow \forall x\ p(x)$

We use the rules for negating quantified statements in the following example.

EXAMPLE 2.44

Here we find the negation of two statements, where the universe comprises all of the integers.

1) Let $p(x)$ and $q(x)$ be given by

$$p(x):\quad x \text{ is odd} \qquad q(x):\quad x^2 - 1 \text{ is even.}$$

The statement "If x is odd, then $x^2 - 1$ is even" can be symbolized as $\forall x\ [p(x) \rightarrow q(x)]$. (This is a true statement.)

The negation of this statement is determined as follows:

$$\begin{aligned}&\neg[\forall x\ (p(x) \rightarrow q(x))] \Longleftrightarrow \exists x\ [\neg(p(x) \rightarrow q(x))]\\ &\Longleftrightarrow \exists x\ [\neg(\neg p(x) \vee q(x))] \Longleftrightarrow \exists x\ [\neg\neg p(x) \wedge \neg q(x)]\\ &\Longleftrightarrow \exists x\ [p(x) \wedge \neg q(x)]\end{aligned}$$

In words, the negation says, "There exists an integer x such that x is odd and $x^2 - 1$ is odd (that is, not even)." (This statement is false.)

2) As in Example 2.42, let $r(x)$ and $s(x)$ be the open statements

$$r(x):\quad 2x + 1 = 5 \qquad s(x):\quad x^2 = 9.$$

The quantified statement $\exists x\ [r(x) \wedge s(x)]$ is false because it asserts the existence of at least one integer a such that $2a + 1 = 5$ ($a = 2$) and $a^2 = 9$ ($a = 3$ or -3). Consequently, its negation

$$\neg[\exists x\ (r(x) \wedge s(x))] \Longleftrightarrow \forall x\ [\neg(r(x) \wedge s(x))] \Longleftrightarrow \forall x\ [\neg r(x) \vee \neg s(x)]$$

is true. This negation may be given in words as "For every integer x, $2x + 1 \neq 5$ or $x^2 \neq 9$."

Because a mathematical statement may involve more than one quantifier, we continue this section by offering some examples and making some observations on these types of statements.

EXAMPLE 2.45

Here we have two real variables x, y, so the universe consists of all real numbers. The commutative law for the addition of real numbers may be expressed by

$$\forall x\ \forall y\ (x + y = y + x).$$

This statement may also be given as

$$\forall y\ \forall x\ (x + y = y + x).$$

Likewise, in the case of the multiplication of real numbers, we may write

$$\forall x\ \forall y\ (xy = yx) \quad \text{or} \quad \forall y\ \forall x\ (xy = yx).$$

These two examples suggest the following general result. If $p(x, y)$ is an open statement in the two variables x, y (with either a prescribed universe for both x and y or one prescribed universe for x and a second for y), then the statements $\forall x\ \forall y\ p(x, y)$ and $\forall y\ \forall x\ p(x, y)$ are logically equivalent — that is, the statement $\forall x\ \forall y\ p(x, y)$ is true (respectively, false) if and only if the statement $\forall y\ \forall x\ p(x, y)$ is true (respectively, false). Hence

$$\forall x\ \forall y\ p(x, y) \Longleftrightarrow \forall y\ \forall x\ p(x, y).$$

EXAMPLE 2.46

When dealing with the associative law for the addition of real numbers, we find that for all real numbers x, y, and z,

$$x + (y + z) = (x + y) + z.$$

Using universal quantifiers (with the universe of all real numbers), we may express this by

$$\forall x\ \forall y\ \forall z\ [x + (y + z) = (x + y) + z] \quad \text{or} \quad \forall y\ \forall x\ \forall z\ [x + (y + z) = (x + y) + z].$$

In fact, there are $3! = 6$ ways to order these three universal quantifiers, and all six of these quantified statements are logically equivalent to one another.

This is actually true for all open statements $p(x, y, z)$, and to shorten the notation, one may write, for example,

$$\forall x, y, z\ \ p(x, y, z) \Longleftrightarrow \forall y, x, z\ \ p(x, y, z) \Longleftrightarrow \forall x, z, y\ \ p(x, y, z),$$

describing the logical equivalence for three of the six statements.

In Examples 2.45 and 2.46 we encountered quantified statements with two and three bound variables — each such variable bound by a universal quantifier. Our next example examines a situation in which there are two bound variables — and this time each of these variables is bound by an existential quantifier.

EXAMPLE 2.47

For the universe of all integers, consider the true statement "There exist integers x, y such that $x + y = 6$." We may represent this in symbolic form by

$$\exists x\ \exists y\ (x + y = 6).$$

If we let $p(x, y)$ denote the open statement "$x + y = 6$," then an equivalent statement can be given by $\exists y\ \exists x\ p(x, y)$.

In general, for any open statement $p(x, y)$ and universe(s) prescribed for the variables x, y,

$$\exists x\ \exists y\ p(x, y) \Longleftrightarrow \exists y\ \exists x\ p(x, y).$$

Similar results follow for statements involving three or more such quantifiers.

When a statement involves both existential and universal quantifiers, however, we must be careful about the order in which the quantifiers are written. Example 2.48 illustrates this case.

EXAMPLE 2.48

We restrict ourselves here to the universe of all integers and let $p(x, y)$ denote the open statement "$x + y = 17$."

1) The statement

$$\forall x\ \exists y\ p(x, y)$$

says that "For every integer x, there exists an integer y such that $x + y = 17$." (We read the quantifiers from left to right.)

This statement is true; once we select *any* x, the integer $y = 17 - x$ does *exist* and $x + y = x + (17 - x) = 17$. But we realize that each value of x gives rise to a different value of y.

2) Now consider the statement

$$\exists y\ \forall x\ p(x, y).$$

This statement is read "There exists an integer y so that for all integers x, $x + y = 17$." This statement is false. Once *an* integer y is selected, the *only* value that x can have (and still satisfy $x + y = 17$) is $17 - y$.

If the statement $\exists y\ \forall x\ p(x, y)$ were true, then every integer (x) would equal $17 - y$ (for some one fixed y). This says, in effect, that all integers are equal!

Consequently, the statements $\forall x\ \exists y\ p(x, y)$ and $\exists y\ \forall x\ p(x, y)$ are generally not logically equivalent.

Translating mathematical statements — be they postulates, definitions, or theorems — into symbolic form can be helpful for two important reasons.

1) Doing so forces us to be very careful and precise about the meanings of statements, the meanings of phrases such as "For all x" and "There exists an x," and the order in which such phrases appear.

2) After we translate a mathematical statement into symbolic form, the rules we have learned should then apply when we want to determine such related statements as the negation or, if appropriate, the contrapositive, converse, or inverse.

Our last two examples illustrate this, and in so doing, extend the results in Table 2.23.

EXAMPLE 2.49

Let $p(x, y)$, $q(x, y)$, and $r(x, y)$ represent three open statements, with replacements for the variables x, y chosen from some prescribed universe(s). What is the negation of the following statement?

$$\forall x\ \exists y\ [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

We find that

$$\begin{aligned}
&\neg[\forall x\ \exists y\ [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]\\
\Longleftrightarrow\ &\exists x\ [\neg\exists y\ [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]]\\
\Longleftrightarrow\ &\exists x\ \forall y\ \neg[(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]\\
\Longleftrightarrow\ &\exists x\ \forall y\ \neg[\neg[p(x, y) \wedge q(x, y)] \vee r(x, y)]\\
\Longleftrightarrow\ &\exists x\ \forall y\ [\neg\neg[p(x, y) \wedge q(x, y)] \wedge \neg r(x, y)]\\
\Longleftrightarrow\ &\exists x\ \forall y\ [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)].
\end{aligned}$$

Now suppose that we are trying to establish the validity of an argument (or a mathematical theorem) for which

$$\forall x\ \exists y\ [(p(x, y) \wedge q(x, y)) \rightarrow r(x, y)]$$

is the conclusion. Should we want to try to prove the result by the method of Proof by Contradiction, we would assume as an additional premise the negation of this conclusion. Consequently, our additional premise would be the statement

$$\exists x\ \forall y\ [(p(x, y) \wedge q(x, y)) \wedge \neg r(x, y)].$$

Finally, we consider how to negate the definition of *limit*, a fundamental concept in calculus.

EXAMPLE 2.50

In calculus, one studies the properties of real-valued functions of a real variable. (Functions will be examined in Chapter 5 of this text.) Among these properties is the existence of limits, and one finds the following definition: Let I be an open interval[†] containing the real number a and suppose the function f is defined throughout I, except possibly at a. We say that f has the *limit* L as x approaches a, and write $\lim_{x \to a} f(x) = L$, if (and only if) for every $\epsilon > 0$ there exists a $\delta > 0$ so that, for all x in I, $(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)$. This can be expressed in symbolic form as

$$\lim_{x \to a} f(x) = L \Longleftrightarrow \forall \epsilon > 0\ \exists \delta > 0\ \forall x\ [(0 < |x - a| < \delta) \rightarrow (|f(x) - L| < \epsilon)].$$

[†]The concept of an open interval is defined at the end of Section 3.1.

[Here the universe comprises the real numbers in the open interval I, except possibly a. Also, the quantifiers $\forall\epsilon > 0$ and $\exists\delta > 0$ now contain some restrictive information.] Then, to negate this definition, we do the following (in which certain steps have been combined):

$$\lim_{x\to a} f(x) \neq L$$
$$\Leftrightarrow \neg[\forall\epsilon > 0\ \exists\delta > 0\ \forall x\ [(0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon)]]$$
$$\Leftrightarrow \exists\epsilon > 0\ \forall\delta > 0\ \exists x\ \neg[(0 < |x - a| < \delta) \to (|f(x) - L| < \epsilon)]$$
$$\Leftrightarrow \exists\epsilon > 0\ \forall\delta > 0\ \exists x\ \neg[\neg(0 < |x - a| < \delta) \vee (|f(x) - L| < \epsilon)]$$
$$\Leftrightarrow \exists\epsilon > 0\ \forall\delta > 0\ \exists x\ [\neg\neg(0 < |x - a| < \delta) \wedge \neg(|f(x) - L| < \epsilon)]$$
$$\Leftrightarrow \exists\epsilon > 0\ \forall\delta > 0\ \exists x\ [(0 < |x - a| < \delta) \wedge (|f(x) - L| \geq \epsilon)]$$

Translating into words, we find that $\lim_{x\to a} f(x) \neq L$ if (and only if) there exists a positive (real) number ϵ such that for every positive (real) number δ, there is an x in I such that $0 < |x - a| < \delta$ (that is, $x \neq a$ and its distance from a is less than δ) but $|f(x) - L| \geq \epsilon$ [that is, the value of $f(x)$ differs from L by at least ϵ].

EXERCISES 2.4

1. Let $p(x)$, $q(x)$ denote the following open statements.

$p(x)$: $x \leq 3$ $\qquad$ $q(x)$: $x + 1$ is odd

If the universe consists of all integers, what are the truth values of the following statements?

a) $q(1)$ **b)** $\neg p(3)$ **c)** $p(7) \vee q(7)$
d) $p(3) \wedge q(4)$ **e)** $\neg(p(-4) \vee q(-3))$
f) $\neg p(-4) \wedge \neg q(-3)$

2. Let $p(x)$, $q(x)$ be defined as in Exercise 1. Let $r(x)$ be the open statement "$x > 0$." Once again the universe comprises all integers.

a) Determine the truth values of the following statements.
- **i)** $p(3) \vee [q(3) \vee \neg r(3)]$
- **ii)** $p(2) \to [q(2) \to r(2)]$
- **iii)** $[p(2) \wedge q(2)] \to r(2)$
- **iv)** $p(0) \to [\neg q(-1) \leftrightarrow r(1)]$

b) Determine all values of x for which $[p(x) \wedge q(x)] \wedge r(x)$ results in a true statement.

3. Let $p(x)$ be the open statement "$x^2 = 2x$," where the universe comprises all integers. Determine whether each of the following statements is true or false.

a) $p(0)$ **b)** $p(1)$ **c)** $p(2)$
d) $p(-2)$ **e)** $\exists x\ p(x)$ **f)** $\forall x\ p(x)$

4. Consider the universe of all polygons with three or four sides, and define the following open statements for this universe.

$a(x)$: all interior angles of x are equal
$e(x)$: x is an equilateral triangle
$h(x)$: all sides of x are equal
$i(x)$: x is an isosceles triangle
$p(x)$: x has an interior angle that exceeds 180°
$q(x)$: x is a quadrilateral
$r(x)$: x is a rectangle
$s(x)$: x is a square
$t(x)$: x is a triangle

Translate each of the following statements into an English sentence, and determine whether the statement is true or false.

a) $\forall x\ [q(x) \veebar t(x)]$ **b)** $\forall x\ [i(x) \to e(x)]$
c) $\exists x\ [t(x) \wedge p(x)]$ **d)** $\forall x\ [(a(x) \wedge t(x)) \leftrightarrow e(x)]$
e) $\exists x\ [q(x) \wedge \neg r(x)]$ **f)** $\exists x\ [r(x) \wedge \neg s(x)]$
g) $\forall x\ [h(x) \to e(x)]$ **h)** $\forall x\ [t(x) \to \neg p(x)]$
i) $\forall x\ [s(x) \leftrightarrow (a(x) \wedge h(x))]$
j) $\forall x\ [t(x) \to (a(x) \leftrightarrow h(x))]$

5. Professor Carlson's class in mechanics is comprised of 29 students of which exactly

1) three physics majors are juniors;
2) two electrical engineering majors are juniors;
3) four mathematics majors are juniors;
4) twelve physics majors are seniors;
5) four electrical engineering majors are seniors;
6) two electrical engineering majors are graduate students; and
7) two mathematics majors are graduate students.

Consider the following open statements.

$c(x)$: Student x is in the class (that is, Professor Carlson's mechanics class as already described).

$j(x)$: Student x is a junior.

$s(x)$: Student x is a senior.

$g(x)$: Student x is a graduate student.

$p(x)$: Student x is a physics major.

$e(x)$: Student x is an electrical engineering major.

$m(x)$: Student x is a mathematics major.

Write each of the following statements in terms of quantifiers and the open statements $c(x)$, $j(x)$, $s(x)$, $g(x)$, $p(x)$, $e(x)$, and $m(x)$, and determine whether the given statement is true or false. Here the universe comprises all of the 12,500 students enrolled at the university where Professor Carlson teaches. Furthermore, at this university each student has only one major.

a) There is a mathematics major in the class who is a junior.

b) There is a senior in the class who is not a mathematics major.

c) Every student in the class is majoring in mathematics or physics.

d) No graduate student in the class is a physics major.

e) Every senior in the class is majoring in either physics or electrical engineering.

6. Let $p(x, y)$, $q(x, y)$ denote the following open statements.

$p(x, y)$: $x^2 \geq y$ $\quad$ $q(x, y)$: $x + 2 < y$

If the universe for each of x, y consists of all real numbers, determine the truth value for each of the following statements.

a) $p(2, 4)$ **b)** $q(1, \pi)$

c) $p(-3, 8) \wedge q(1, 3)$ **d)** $p\left(\frac{1}{2}, \frac{1}{3}\right) \vee \neg q(-2, -3)$

e) $p(2, 2) \rightarrow q(1, 1)$ **f)** $p(1, 2) \leftrightarrow \neg q(1, 2)$

7. For the universe of all integers, let $p(x)$, $q(x)$, $r(x)$, $s(x)$, and $t(x)$ be the following open statements.

$p(x)$: $x > 0$

$q(x)$: x is even

$r(x)$: x is a perfect square

$s(x)$: x is (exactly) divisible by 4

$t(x)$: x is (exactly) divisible by 5

a) Write the following statements in symbolic form.

i) At least one integer is even.
ii) There exists a positive integer that is even.
iii) If x is even, then x is not divisible by 5.
iv) No even integer is divisible by 5.
v) There exists an even integer divisible by 5.
vi) If x is even and x is a perfect square, then x is divisible by 4.

b) Determine whether each of the six statements in part (a) is true or false. For each false statement, provide a counterexample.

c) Express each of the following symbolic representations in words.

i) $\forall x\, [r(x) \rightarrow p(x)]$ **ii)** $\forall x\, [s(x) \rightarrow q(x)]$
iii) $\forall x\, [s(x) \rightarrow \neg t(x)]$ **iv)** $\exists x\, [s(x) \wedge \neg r(x)]$

d) Provide a counterexample for each false statement in part (c).

8. Let $p(x)$, $q(x)$, and $r(x)$ denote the following open statements.

$p(x)$: $x^2 - 8x + 15 = 0$

$q(x)$: x is odd

$r(x)$: $x > 0$

For the universe of all integers, determine the truth or falsity of each of the following statements. If a statement is false, give a counterexample.

a) $\forall x\, [p(x) \rightarrow q(x)]$ **b)** $\forall x\, [q(x) \rightarrow p(x)]$

c) $\exists x\, [p(x) \rightarrow q(x)]$ **d)** $\exists x\, [q(x) \rightarrow p(x)]$

e) $\exists x\, [r(x) \rightarrow p(x)]$ **f)** $\forall x\, [\neg q(x) \rightarrow \neg p(x)]$

g) $\exists x\, [p(x) \rightarrow (q(x) \wedge r(x))]$

h) $\forall x\, [(p(x) \vee q(x)) \rightarrow r(x)]$

9. Let $p(x)$, $q(x)$, and $r(x)$ be the following open statements.

$p(x)$: $x^2 - 7x + 10 = 0$

$q(x)$: $x^2 - 2x - 3 = 0$

$r(x)$: $x < 0$

a) Determine the truth or falsity of the following statements, where the universe is all integers. If a statement is false, provide a counterexample or explanation.

i) $\forall x\, [p(x) \rightarrow \neg r(x)]$ **ii)** $\forall x\, [q(x) \rightarrow r(x)]$
iii) $\exists x\, [q(x) \rightarrow r(x)]$ **iv)** $\exists x\, [p(x) \rightarrow r(x)]$

b) Find the answers to part (a) when the universe consists of all positive integers.

c) Find the answers to part (a) when the universe contains only the integers 2 and 5.

10. For the following program segment, m and n are integer variables. The variable A is a two-dimensional array $A[1, 1]$, $A[1, 2]$, ..., $A[1, 20]$, ..., $A[10, 1]$, ..., $A[10, 20]$, with 10 rows (indexed from 1 to 10) and 20 columns (indexed from 1 to 20).

```
for m := 1 to 10 do
  for n := 1 to 20 do
    A[m,n] := m + 3 * n
```

Write the following statements in symbolic form. (The universe for the variable m contains only the integers from 1 to 10 inclusive; for n the universe consists of the integers from 1 to 20 inclusive.)

a) All entries of A are positive.

b) All entries of A are positive and less than or equal to 70.

c) Some of the entries of A exceed 60.

d) The entries in each row of A are sorted into (strictly) ascending order.

e) The entries in each column of A are sorted into (strictly) ascending order.

f) The entries in the first three rows of A are distinct.

11. Identify the bound variables and the free variables in each of the following expressions (or statements). In both cases the universe comprises all real numbers.

a) $\forall y\ \exists z\ [\cos(x+y) = \sin(z-x)]$

b) $\exists x\ \exists y\ [x^2 - y^2 = z]$

12. a) Let $p(x, y)$ denote the open statement "x divides y," where the universe for each of the variables x, y comprises all integers. (In this context "divides" means "exactly divides" or "divides evenly.") Determine the truth value of each of the following statements; if a quantified statement is false, provide an explanation or a counterexample.

i) $p(3, 7)$ **ii)** $p(3, 27)$
iii) $\forall y\ p(1, y)$ **iv)** $\forall x\ p(x, 0)$
v) $\forall x\ p(x, x)$ **vi)** $\forall y\ \exists x\ p(x, y)$
vii) $\exists y\ \forall x\ p(x, y)$
viii) $\forall x\ \forall y\ [(p(x, y) \wedge p(y, x)) \rightarrow (x = y)]$

b) Determine which of the eight statements in part (a) will change in truth value if the universe for each of the variables x, y were restricted to just the positive integers.

c) Determine the truth value of each of the following statements. If the statement is false, provide an explanation or a counterexample. [The universe for each of x, y is as in part (b).]

i) $\forall x\ \exists y\ p(x, y)$ **ii)** $\forall y\ \exists x\ p(x, y)$
iii) $\exists x\ \forall y\ p(x, y)$ **iv)** $\exists y\ \forall x\ p(x, y)$

13. Suppose that $p(x, y)$ is an open statement where the universe for each of x, y consists of only three integers: 2, 3, 5. Then the quantified statement $\exists y\ p(2, y)$ is logically equivalent to $p(2, 2) \vee p(2, 3) \vee p(2, 5)$. The quantified statement $\exists x\ \forall y\ p(x, y)$ is logically equivalent to $[p(2, 2) \wedge p(2, 3) \wedge p(2, 5)] \vee [p(3, 2) \wedge p(3, 3) \wedge p(3, 5)] \vee [p(5, 2) \wedge p(5, 3) \wedge p(5, 5)]$. Use conjunctions and/or disjunctions to express the following statements without quantifiers.

a) $\forall x\ p(x, 3)$ **b)** $\exists x\ \exists y\ p(x, y)$ **c)** $\forall y\ \exists x\ p(x, y)$

14. Let $p(n), q(n)$ represent the open statements

$p(n)$: n is odd $\qquad q(n)$: n^2 is odd

for the universe of all integers. Which of the following statements are logically equivalent to each other?

a) If the square of an integer is odd, then the integer is odd.

b) $\forall n\ [p(n)$ is necessary for $q(n)]$

c) The square of an odd integer is odd.

d) There are some integers whose squares are odd.

e) Given an integer whose square is odd, that integer is likewise odd.

f) $\forall n\ [\neg p(x) \rightarrow \neg q(n)]$

g) $\forall n\ [p(n)$ is sufficient for $q(n)]$

15. For each of the following pairs of statements determine whether the proposed negation is correct. If correct, determine which is true: the original statement or the proposed negation. If the proposed negation is wrong, write a correct version of the negation and then determine whether the original statement or your corrected version of the negation is true.

a) Statement: For all real numbers x, y, if $x^2 > y^2$, then $x > y$.
Proposed negation: There exist real numbers x, y such that $x^2 > y^2$ but $x \leq y$.

b) Statement: There exist real numbers x, y such that x and y are rational but $x + y$ is irrational.
Proposed negation: For all real numbers x, y, if $x + y$ is rational, then each of x, y is rational.

c) Statement: For all real numbers x, if x is not 0, then x has a multiplicative inverse.
Proposed negation: There exists a nonzero real number that does not have a multiplicative inverse.

d) Statement: There exist odd integers whose product is odd.
Proposed negation: The product of any two odd integers is odd.

16. Write the negation of each of the following statements as an English sentence—without symbolic notation. (Here the universe consists of all the students at the university where Professor Lenhart teaches.)

a) Every student in Professor Lenhart's C++ class is majoring in computer science or mathematics.

b) At least one student in Professor Lenhart's C++ class is a history major.

17. Write the negation of each of the following true statements. For parts (a) and (b) the universe consists of all integers; for parts (c) and (d) the universe comprises all real numbers.

a) For all integers n, if n is not (exactly) divisible by 2, then n is odd.

b) If k, m, n are any integers where $k - m$ and $m - n$ are odd, then $k - n$ is even.

c) If x is a real number where $x^2 > 16$, then $x < -4$ or $x > 4$.

d) For all real numbers x, if $|x - 3| < 7$, then $-4 < x < 10$.

18. Negate and simplify each of the following.

a) $\exists x\ [p(x) \vee q(x)]$ **b)** $\forall x\ [p(x) \wedge \neg q(x)]$

c) $\forall x\ [p(x) \rightarrow q(x)]$

d) $\exists x\ [(p(x) \vee q(x)) \rightarrow r(x)]$

19. For each of the following statements state the converse, inverse, and contrapositive. Also determine the truth value for each given statement, as well as the truth values for its converse, inverse, and contrapositive. (Here "divides" means "exactly divides.")

a) [The universe comprises all positive integers.]
If $m > n$, then $m^2 > n^2$.

b) [The universe comprises all integers.]
If $a > b$, then $a^2 > b^2$.

c) [The universe comprises all integers.]
If m divides n and n divides p, then m divides p.

d) [The universe consists of all real numbers.]
$\forall x\, [(x > 3) \rightarrow (x^2 > 9)]$

e) [The universe consists of all real numbers.]
For all real numbers x, if $x^2 + 4x - 21 > 0$, then $x > 3$ or $x < -7$.

20. Rewrite each of the following statements in the *if-then* form. Then write the converse, inverse, and contrapositive of your implication. For each result in parts (a) and (c) give the truth value for the implication and the truth values for its converse, inverse, and contrapositive. [In part (a) "divisibility" requires a remainder of 0.]

a) [The universe comprises all positive integers.]
Divisibility by 21 is a sufficient condition for divisibility by 7.

b) [The universe comprises all snakes presently slithering about the jungles of Asia.]
Being a cobra is a sufficient condition for a snake to be dangerous.

c) [The universe consists of all complex numbers.]
For every complex number z, z being real is necessary for z^2 to be real.

21. For the following statements the universe comprises all nonzero integers. Determine the truth value of each statement.

a) $\exists x\, \exists y\, [xy = 1]$ **b)** $\exists x\, \forall y\, [xy = 1]$

c) $\forall x\, \exists y\, [xy = 1]$

d) $\exists x\, \exists y\, [(2x + y = 5) \wedge (x - 3y = -8)]$

e) $\exists x\, \exists y\, [(3x - y = 7) \wedge (2x + 4y = 3)]$

22. Answer Exercise 21 for the universe of all nonzero real numbers.

23. In the arithmetic of real numbers, there is a real number, namely 0, called the identity of addition because $a + 0 = 0 + a = a$ for every real number a. This may be expressed in symbolic form by

$$\exists z\, \forall a\, [a + z = z + a = a].$$

(We agree that the universe comprises all real numbers.)

a) In conjunction with the existence of an additive identity is the existence of additive inverses. Write a quantified statement that expresses "Every real number has an additive inverse." (Do not use the minus sign anywhere in your statement.)

b) Write a quantified statement dealing with the existence of a multiplicative identity for the arithmetic of real numbers.

c) Write a quantified statement covering the existence of multiplicative inverses for the nonzero real numbers. (Do not use the exponent -1 anywhere in your statement.)

d) Do the results in parts (b) and (c) change in any way when the universe is restricted to the integers?

24. Consider the quantified statement $\forall x\, \exists y\, [x + y = 17]$. Determine whether this statement is true or false for each of the following universes: (a) the integers; (b) the positive integers; (c) the integers for x, the positive integers for y; (d) the positive integers for x, the integers for y.

25. Let the universe for the variables in the following statements consist of all real numbers. In each case negate and simplify the given statement.

a) $\forall x\, \forall y\, [(x > y) \rightarrow (x - y > 0)]$

b) $\forall x\, \forall y\, [(x < y) \rightarrow \exists z\, (x < z < y)]$

c) $\forall x\, \forall y\, [(|x| = |y|) \rightarrow (y = \pm x)]$

26. In calculus the definition of the limit L of a sequence of real numbers $r_1, r_2, r_3, \ldots$ can be given as

$$\lim_{n \to \infty} r_n = L$$

if (and only if) for every $\epsilon > 0$ there exists a positive integer k so that for all integers n, if $n > k$ then $|r_n - L| < \epsilon$.

In symbolic form this can be expressed as

$$\lim_{n \to \infty} r_n = L \Longleftrightarrow \forall \epsilon > 0\, \exists k > 0\, \forall n\, [(n > k) \rightarrow |r_n - L| < \epsilon].$$

Express $\lim_{n \to \infty} r_n \neq L$ in symbolic form.

2.5 Quantifiers, Definitions, and the Proofs of Theorems

In this section we shall combine some of the ideas we have already studied in the prior two sections. Although Section 2.3 introduced rules and methods for establishing the validity of an argument, unfortunately the arguments presented there seemed to have little to do with anything mathematical. [The rare exceptions are in Example 2.23 and the erroneous argument in part (b) of the material preceding Example 2.26.] Most of the arguments dealt with certain individuals and predicaments they were either in or about to face.

But now that we have learned some of the properties of quantifiers and quantified statements, we are better equipped to handle arguments that will help us to prove mathematical theorems. Before dealing with theorems, however, we shall consider how mathematical definitions are traditionally presented in scientific writing.

Following Example 2.3 in Section 2.1, the discussion concerned how an implication might be used in place of a biconditional in everyday conversation. But in scientific writing, it was noted, we should avoid any and all situations where an ambiguous interpretation might come about — in particular, an implication should not be used when a biconditional is intended. However, there is one major exception to that rule and it concerns the way that mathematical definitions are traditionally presented in mathematics textbooks and other scientific literature. Example 2.51 demonstrates this exception.

EXAMPLE 2.51

a) Let us start with the universe of all quadrilaterals in the plane and try to identify those that are called rectangles.

One person might say that

"If a quadrilateral is a rectangle then it has four equal angles."

Another individual might identify these special quadrilaterals by observing that

"If a quadrilateral has four equal angles, then it is a rectangle."

(Here both people are making implicitly quantified statements, where the quantifier is universal.)

Given the open statements

$p(x)$: x is a rectangle $\qquad q(x)$: x has four equal angles,

we can express what the first person says as

$$\forall x\ [p(x) \rightarrow q(x)],$$

while for the second person we would write

$$\forall x\ [q(x) \rightarrow p(x)].$$

So which of the preceding (quantified) statements identifies or defines a rectangle? Perhaps we feel that they both do. But how can that be, since one statement is the converse of the other and, in general, the converse of an implication is *not* logically equivalent to the implication.

Here the reader must consider what is intended — not just what each of the two people has said, or the symbolic expressions we have written to represent these statements. In this situation each person is using an implication with the meaning of a biconditional. They are both intending (though not stating)

$$\forall x\ [p(x) \leftrightarrow q(x)],$$

— that is, each is really telling us that

"A quadrilateral is a rectangle *if and only if* it has four equal angles."

b) Within the universe of all integers we can distinguish the even integers by means of a certain property and so we may define them as follows:

For every integer n we call n even if it is divisible by 2.

(By the expression "divisible by 2" we mean "exactly divisible by 2" — that is, there is no remainder upon division of the dividend n by the divisor 2.)

If we consider the open statements

$$p(n): \quad n \text{ is an even integer} \qquad q(n): \quad n \text{ is divisible by 2,}$$

then it *appears* that the preceding definition may be written symbolically as

$$\forall n\ [q(n) \rightarrow p(n)].$$

After all, the given quantified statement (in the preceding definition) is an implication. However, the situation here is similar to that given in part (a). What appears to be stated is not what is intended. The intention is for the reader to interpret the given definition as

$$\forall n\ [q(n) \leftrightarrow p(n)],$$

that is,

"For every integer n, we call n even *if and only if* n is divisible by 2."

(Note that the open statement "n is divisible by 2" can also be expressed by the *open* statement "$n = 2k$, for some integer k." Don't be misled here by the presence of the quantifier "for some integer k" — for the expression $\exists k\ [n = 2k]$ is still an open statement because n remains a free variable.)

So now we see how quantifiers may enter into the way we state mathematical definitions — and that the traditional way in which such a definition appears is as an implication. But beware and remember: It is *only in definitions* that an implication can be (mis)read and correctly interpreted as a biconditional.

Note how we defined the limit concept in Example 2.50. There we wrote "if (and only if)" since we wanted to let the reader know our intention. Now we are free to replace "if (and only if)" by simply "if."

Having settled our discussion on the nature of mathematical definitions, we continue now with an investigation of arguments involving quantified statements.

EXAMPLE 2.52

Suppose that we start with the universe that comprises only the 13 integers 2, 4, 6, 8, . . . , 24, 26. Then we can establish the statement:

For all n (meaning $n = 2, 4, 6, \ldots, 26$),

we can write n as the sum of at most three perfect squares.

The results in Table 2.24 provide a case-by-case verification showing the given (quantified) statement to be true. (We might call this statement a theorem.)

Table 2.24

$2 = 1 + 1$	$10 = 9 + 1$	$20 = 16 + 4$
$4 = 4$	$12 = 4 + 4 + 4$	$22 = 9 + 9 + 4$
$6 = 4 + 1 + 1$	$14 = 9 + 4 + 1$	$24 = 16 + 4 + 4$
$8 = 4 + 4$	$16 = 16$	$26 = 25 + 1$
	$18 = 16 + 1 + 1$	

This exhaustive listing is an example of a proof using the technique we call, rather appropriately, the *method of exhaustion*. This method is reasonable when we are dealing with a fairly small universe. If we are confronted with a situation in which the universe is larger but within the range of a computer that is available to us, then we might write a program to check all of the individual cases.

(Note that for certain cases in Table 2.24 more than one answer may be possible. For example, we could have written $18 = 9 + 9$ and $26 = 16 + 9 + 1$. But this is all right. We were told that each positive even integer less than or equal to 26 could be written as the sum of one, two, or three perfect squares. We were *not* told that each such representation had to be unique, so more than one possibility could occur. What we had to check in each case was that there was at least one possibility.)

In the previous example we mentioned the word *theorem*. We also found this term used in Chapter 1 — for example, in results like the binomial theorem and the multinomial theorem where we were introduced to certain types of enumeration problems. Without getting overly technical, we shall consider *theorems* to be statements of mathematical interest, statements that are known to be true. Sometimes the term *theorem* is used only to describe major results that have many and varied consequences. Certain of these consequences that follow rather immediately from a theorem are termed *corollaries* (as in the case of Corollary 1.1 in Section 1.3). In this text, however, we shall not be so particular in our use of the word theorem.

Example 2.52 is a nice starting point to examine the proof of a quantified statement. Unfortunately, a great number of mathematical statements and theorems often deal with universes that do not lend themselves to the method of exhaustion. When faced with establishing or proving a result for all integers, for example, or for all real numbers, then we cannot use a case-by-case method like the one in Example 2.52. So what can we do?

We start by considering the following rule.

The Rule of Universal Specification: If an open statement becomes true for *all* replacements by the members in a given universe, then that open statement is true for *each specific* individual member in that universe. (A bit more symbolically — if $p(x)$ is an open statement for a given universe, and if $\forall x\ p(x)$ is true, then $p(a)$ is true for each a in the universe.)

This rule indicates that the truth of an open statement in one particular instance follows (as a special case) from the more general (for the entire universe) truth of that universally quantified open statement. The following examples will show us how to apply this idea.

EXAMPLE 2.53

a) For the universe of all people, consider the open statements

$m(x)$: x is a mathematics professor $\qquad$ $c(x)$: x has studied calculus.

Now consider the following argument.

All mathematics professors have studied calculus.

Leona is a mathematics professor.

Therefore Leona has studied calculus.

If we let l represent this particular woman (in our universe) named Leona, then we can rewrite this argument in symbolic form as

$$\begin{array}{l} \forall x\ [m(x) \rightarrow c(x)] \\ m(l) \\ \hline \therefore c(l) \end{array}$$

Here the two statements above the line are the premises of the argument, and the statement $c(l)$ below the line is its conclusion. This is comparable to what we saw in Section 2.3, except now we have a premise that is a universally quantified statement. As was the case in Section 2.3, the premises are all assumed to be true and we must try to establish that the conclusion is also true under these circumstances. Now, to establish the validity of the given argument, we proceed as follows.

Steps	**Reasons**
1) $\forall x\ [m(x) \rightarrow c(x)]$	Premise
2) $m(l)$	Premise
3) $m(l) \rightarrow c(l)$	Step (1) and the Rule of Universal Specification
4) $\therefore c(l)$	Steps (2) and (3) and the Rule of Detachment

Note that the statements in steps (2) and (3) are *not* quantified statements. They are the types of statements we studied earlier in the chapter. In particular, we can apply the rules of inference we learned in Section 2.3 to these two statements to deduce the conclusion in step (4).

We see here that the Rule of Universal Specification enables us to take a universally quantified premise and deduce from it an ordinary statement (that is, one that is not quantified). This (ordinary) statement — namely, $m(l) \rightarrow c(l)$ — is one specific true instance of the universally quantified true premise $\forall x\ [m(x) \rightarrow c(x)]$.

b) For an example of a more mathematical nature let us consider the universe of all triangles in the plane in conjunction with the open statements

$p(t)$: t has two sides of equal length.

$q(t)$: t is an isosceles triangle.

$r(t)$: t has two angles of equal measure.

Let us also focus our attention on one specific triangle with no two angles of equal measure. This triangle will be called triangle XYZ and will be designated by c. Then we find that the argument

In triangle XYZ there is no pair of angles of equal measure.	$\neg r(c)$
If a triangle has two sides of equal length, then it is isosceles.	$\forall t\ [p(t) \rightarrow q(t)]$
If a triangle is isosceles, then it has two angles of equal measure.	$\forall t\ [q(t) \rightarrow r(t)]$
Therefore triangle XYZ has no two sides of equal length.	$\therefore \neg p(c)$

is a valid one — as evidenced by the following.

Steps	**Reasons**
1) $\forall t\,[p(t) \rightarrow q(t)]$	Premise
2) $p(c) \rightarrow q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall t\,[q(t) \rightarrow r(t)]$	Premise
4) $q(c) \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\neg r(c)$	Premise
7) $\therefore \neg p(c)$	Steps (5) and (6) and Modus Tollens

Once again we see how the Rule of Universal Specification helps us. Here it has taken the universally quantified statements at steps (1) and (3) and has provided us with the (ordinary) statements at steps (2) and (4), respectively. Then at this point we were able to apply the rules of inference we learned in Section 2.3 (namely, the Law of the Syllogism and Modus Tollens) to derive the conclusion $\neg p(c)$ in step (7).

c) Now for one last argument to drive the point home! Here we'll consider the universe to be made up of the entire student body at a particular college. One specific student, Mary Gusberti, will be designated by m.

For this universe and the open statements

$j(x)$: x is a junior $\qquad$ $s(x)$: x is a senior

$p(x)$: x is enrolled in a physical education class

we consider the argument:

No junior or senior is enrolled in a physical education class.

Mary Gusberti is enrolled in a physical education class.

Therefore Mary Gusberti is not a senior.

In symbolic form this argument becomes

$$\begin{array}{l} \forall x\,[(j(x) \vee s(x)) \rightarrow \neg p(x)] \\ \underline{p(m) \qquad\qquad\qquad\qquad} \\ \therefore \neg s(m) \end{array}$$

Now the following steps (and reasons) establish the validity of this argument.

Steps	**Reasons**
1) $\forall x\,[(j(x) \vee s(x)) \rightarrow \neg p(x)]$	Premise
2) $p(m)$	Premise
3) $(j(m) \vee s(m)) \rightarrow \neg p(m)$	Step (1) and the Rule of Universal Specification
4) $p(m) \rightarrow \neg(j(m) \vee s(m))$	Step (3), $(q \rightarrow t) \Longleftrightarrow (\neg t \rightarrow \neg q)$, and the Law of Double Negation
5) $p(m) \rightarrow (\neg j(m) \wedge \neg s(m))$	Step (4) and DeMorgan's Law
6) $\neg j(m) \wedge \neg s(m)$	Steps (2) and (5) and the Rule of Detachment (or Modus Ponens)
7) $\therefore \neg s(m)$	Step (6) and the Rule of Conjunctive Simplification

In Example 2.53 we have had our first opportunity to apply the Rule of Universal Specification. Using the rule in conjunction with Modus Ponens (or the Rule of Detachment) and

Modus Tollens, we are able to state the following corresponding analogs, each of which involves a universally quantified premise. In either case we consider a fixed universe that includes a specific member c and make use of the open statements $p(x)$, $q(x)$ defined for this universe.

$$(1)\quad \begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \underline{p(c)\qquad\qquad\quad} \\ \therefore q(c) \end{array} \qquad\qquad (2)\quad \begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \underline{\neg q(c)\qquad\qquad\quad} \\ \therefore \neg p(c) \end{array}$$

These two valid arguments are presented here for the same reason we presented them for the rules of inference — Modus Ponens and Modus Tollens — in Section 2.3 (in the discussion between Examples 2.25 and 2.26). We want to examine some possible errors that may arise when the results in (1) and (2) are not used correctly.

Let us start with the universe of all polygons in the plane. Within this universe we shall let c denote one specific polygon — the quadrilateral $EFGH$, where the measure of angle E is 91°. For the open statements

$$p(x):\quad x \text{ is a square} \qquad\qquad q(x):\quad x \text{ has four sides,}$$

the following argument is *invalid*.

(1′) All squares have four sides.
Quadrilateral $EFGH$ has four sides.
Therefore quadrilateral $EFGH$ is a square.

In symbolic form this argument translates into

$$(1'')\quad \begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \underline{q(c)\qquad\qquad\quad} \\ \therefore p(c) \end{array}$$

Unfortunately, although the premises are true, the conclusion is false. (For a square has no angle of measure 91°.) We admit that there might be some confusion between this argument and the valid one in (1) above. For when we apply the Rule of Universal Specification to the quantified premise in (1″), in this instance we arrive at the *invalid* argument

$$\begin{array}{l} p(c) \rightarrow q(c) \\ \underline{q(c)\qquad\quad} \\ \therefore p(c) \end{array}$$

And here, as in Section 2.3, the error in reasoning lies in our attempt to argue by the converse.

A second invalid argument — from the misuse of argument (2) above — can also be given, as shown in the following.

(2′) All squares have four sides.
Quadrilateral $EFGH$ is not a square.
Therefore quadrilateral $EFGH$ does not have four sides.

Translating (2′) into symbolic form results in

$$(2'')\quad \begin{array}{l} \forall x\,[p(x) \rightarrow q(x)] \\ \underline{\neg p(c)\qquad\qquad\quad} \\ \therefore \neg q(c) \end{array}$$

This time the Rule of Universal Specification leads us to

$$\begin{array}{l} p(c) \rightarrow q(c) \\ \underline{\neg p(c) \qquad\quad} \\ \therefore \neg q(c) \end{array}$$

where the fallacy arises because we are trying to argue by the inverse.

And now let us look back at the three parts of Example 2.53. Although the arguments presented there involved premises that were universally quantified statements, there was never any instance where a universally quantified statement appeared in the conclusion. We now want to remedy this situation, since many theorems in mathematics have the form of a universally quantified statement. To do so we need the following considerations.

Start with a given universe and the open statement $p(x)$. To establish the truth of the statement $\forall x\ p(x)$, we must establish the truth of $p(c)$ for each member c in the given universe. But if the universe has many members or, for example, contains all the positive integers, then this exhaustive, if not exhausting, task of validating the truth of each $p(c)$ becomes difficult, if not impossible. To get around this situation we shall prove that $p(c)$ is true — but now we do it for the case where c denotes a *specific but arbitrarily chosen* member from the prescribed universe.

Should the preceding open statement $p(x)$ have the form $q(x) \rightarrow r(x)$, for open statements $q(x)$ and $r(x)$, then we shall assume the truth of $q(c)$ as an additional premise and try to deduce the truth of $r(c)$ — by using definitions, axioms, previously proven theorems, and the logical principles we have studied. For when $q(c)$ is false, the implication $q(c) \rightarrow r(c)$ is true, regardless of the truth value of $r(c)$.

The reason that the element c must be arbitrary (or generic) is to make sure that what we do and prove about c is applicable for *all* the other elements in the universe. If we are dealing with the universe of all integers, for example, we cannot choose c in an arbitrary manner by selecting c as 4, or by selecting c as an even integer. In general, we cannot make any assumptions about our choice for c unless these assumptions are valid for *all* the other elements of the universe. The word *generic* is applied to the element c here because it indicates that our choice (for c) must share all of the common characteristics of the elements for the given universe.

The principle we have described in the preceding three paragraphs is named and summarized as follows.

The Rule of Universal Generalization: If an open statement $p(x)$ is proved to be true when x is replaced by any *arbitrarily chosen* element c from our universe, then the universally quantified statement $\forall x\ p(x)$ is true. Furthermore, the rule extends beyond a single variable. So if, for example, we have an open statement $q(x, y)$ that is proved to be true when x and y are replaced by *arbitrarily chosen* elements from the same universe, or their own respective universes, then the universally quantified statement $\forall x\ \forall y\ q(x, y)$ [or, $\forall x, y\ q(x, y)$] is true. Similar results hold for the cases of three or more variables.

Before we demonstrate the use of this rule in any examples, we wish to look back at part (1) of Example 2.43 in Section 2.4. It turns out that the explanation given there to establish that

$$\forall x\ [p(x) \wedge (q(x) \wedge r(x))] \Longleftrightarrow \forall x\ [(p(x) \wedge q(x)) \wedge r(x)]$$

anticipated what we have now described in detail as the Rules of Universal Specification and Universal Generalization.

Now we'll turn to an example which is strictly symbolic. This example provides an opportunity to apply the Rule of Universal Generalization.

EXAMPLE 2.54

Let $p(x)$, $q(x)$, and $r(x)$ be open statements that are defined for a given universe. We show that the argument

$$\begin{array}{r} \forall x\,[p(x) \rightarrow q(x)] \\ \underline{\forall x\,[q(x) \rightarrow r(x)]} \\ \therefore \forall x\,[p(x) \rightarrow r(x)] \end{array}$$

is valid by considering the following.

Steps	**Reasons**
1) $\forall x\,[(p(x) \rightarrow q(x)]$	Premise
2) $p(c) \rightarrow q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x\,[q(x) \rightarrow r(x)]$	Premise
4) $q(c) \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $p(c) \rightarrow r(c)$	Steps (2) and (4) and the Law of the Syllogism
6) $\therefore \forall x\,[p(x) \rightarrow r(x)]$	Step (5) and the Rule of Universal Generalization

Here the element c introduced in steps (2) and (4) is the same *specific* but *arbitrarily chosen* element from the universe. Since this element c has no *special or distinguishing properties* but does share all of the common features of every other element in this universe, we can use the Rule of Universal Generalization to go from step (5) to step (6).

And so at last we have dealt with a valid argument where a universally quantified statement appears as the conclusion, as well as among the premises.

The question that now may be at the back of the reader's mind is one of practicality. Namely, when would we ever need to use the argument that we validated in Example 2.54? We may find that we have already used it (perhaps, unknowingly) in earlier algebra and geometry courses, as we demonstrate in the following example.

EXAMPLE 2.55

a) For the universe of all real numbers, consider the open statements

$$p(x):\quad 3x - 7 = 20 \qquad q(x):\quad 3x = 27 \qquad r(x):\quad x = 9.$$

The following solution of an algebraic equation parallels the valid argument from Example 2.54.

1) If $3x - 7 = 20$, then $3x = 27$.	$\forall x\,[p(x) \rightarrow q(x)]$
2) If $3x = 27$, then $x = 9$.	$\underline{\forall x\,[q(x) \rightarrow r(x)]}$
3) Therefore, if $3x - 7 = 20$, then $x = 9$.	$\therefore \forall x\,[p(x) \rightarrow r(x)]$

b) When we dealt with the universe of all quadrilaterals in plane geometry, we may have found ourselves relating something like this:

"Since every square is a rectangle, and every rectangle
is a parallelogram, it follows that every square is a parallelogram."

In this case we are using the argument in Example 2.54 for the open statements

$p(x)$: x is a square $\qquad q(x)$: x is a rectangle $\qquad r(x)$: x is a parallelogram.

Now we continue with one more argument to validate.

EXAMPLE 2.56

The steps and reasons needed to establish the validity of the argument

$$\begin{array}{l} \forall x\,[p(x) \vee q(x)] \\ \underline{\forall x\,[(\neg p(x) \wedge q(x)) \rightarrow r(x)]} \\ \therefore \forall x\,[\neg r(x) \rightarrow p(x)] \end{array}$$

are given as follows. [Here the element c is in the universe assigned for the argument. Also, since the conclusion is a universally quantified implication, we can assume $\neg r(c)$ as an additional premise — as was mentioned earlier when the Rule of Universal Generalization was first introduced.]

Steps	Reasons
1) $\forall x\,[p(x) \vee q(x)]$	Premise
2) $p(c) \vee q(c)$	Step (1) and the Rule of Universal Specification
3) $\forall x\,[(\neg p(x) \wedge q(x)) \rightarrow r(x)]$	Premise
4) $[\neg p(c) \wedge q(c)] \rightarrow r(c)$	Step (3) and the Rule of Universal Specification
5) $\neg r(c) \rightarrow \neg[\neg p(c) \wedge q(c)]$	Step (4) and $s \rightarrow t \Longleftrightarrow \neg t \rightarrow \neg s$
6) $\neg r(c) \rightarrow [p(c) \vee \neg q(c)]$	Step (5), DeMorgan's Law, and the Law of Double Negation
7) $\neg r(c)$	Premise (assumed)
8) $p(c) \vee \neg q(c)$	Steps (7) and (6) and Modus Ponens
9) $[p(c) \vee q(c)] \wedge [p(c) \vee \neg q(c)]$	Steps (2) and (8) and the Rule of Conjunction
10) $p(c) \vee [q(c) \wedge \neg q(c)]$	Step (9) and the Distributive Law of $\vee$ over $\wedge$
11) $p(c)$	Step (10), $q(c) \wedge \neg q(c) \Longleftrightarrow F_0$, and $p(c) \vee F_0 \Longleftrightarrow p(c)$
12) $\therefore \forall x\,[\neg r(x) \rightarrow p(x)]$	Steps (7) and (11) and the Rule of Universal Generalization

Before going on we want to point out a convention that the reader may not like but will have to get used to. It concerns our coverage of the Rules of Universal Specification and Universal Generalization. In the first case we started with the statement $\forall x\ p(x)$ and then dealt with $p(c)$ for some specific element c in our universe. For the Rule of Universal Generalization we obtained the truth of $\forall x\ p(x)$ from that of $p(c)$, where c was arbitrarily selected from the universe. Unfortunately, we'll often find ourselves using the letter x instead of c to denote the element — but as long as we understand what is happening we shall soon find the convention easy enough to work with.

The results of Example 2.54 and especially Example 2.56 lead us to believe that we can use universally quantified statements and the rules of inference — including the Rules of Universal Specification and Universal Generalization — to formalize and prove a variety of arguments and, hopefully, theorems. When we do so it appears that the validation of some rather short arguments requires quite a number of steps, because we have been very meticulous and included all the steps and reasons — we left little, if anything, to the imagination. The reader should rest assured that when we start to prove mathematical theorems, we shall present the proofs in the more conventional paragraph style. We shall no longer mention each and every application of the laws of logic and the other tautologies or the rules of inference. On occasion we may single out a certain rule of inference, but our attention will be primarily directed to the use of definitions, mathematical axioms and principles (other than those we have found in our study of logic), and other (earlier) theorems we have been able to prove. Why then have we been learning all of this material on validating arguments? Because it will provide us with a framework to fall back on whenever we doubt whether a given attempt at a proof really does the job. If in doubt, we have our study of logic to supply us with a somewhat mechanical but strictly objective means to help us decide.

And now we present paragraph-style proofs for some results about the integers. (These results may be considered rather obvious to us — in fact, we may find we have already seen and used some of them. But they provide an excellent setting for writing some simple proofs.) The proofs we shall presently introduce use the following ideas, which we now formally define. [The first idea was mentioned earlier in part (b) of Example 2.51.]

Definition 2.8 Let n be an integer. We call n *even* if n is divisible by 2 — that is, if there exists an integer r so that $n = 2r$. If n is not even, then we call n *odd* and find for this case that there exists an integer s where $n = 2s + 1$.

THEOREM 2.2 For all integers k and l, if k, l are both odd, then $k + l$ is even.

Proof: In this proof we shall number the steps so that we may refer to them for some later remarks. After this we shall no longer number the steps.

1) Since k and l are odd, we may write $k = 2a + 1$ and $l = 2b + 1$, for some integers a, b. This is due to Definition 2.8.
2) Then

$$k + l = (2a + 1) + (2b + 1) = 2(a + b + 1),$$

by virtue of the Commutative and Associative Laws of Addition and the Distributive Law of Multiplication over Addition — all of which hold for integers.
3) Since a, b are integers, $a + b + 1 = c$ is an integer; with $k + l = 2c$, it follows from Definition 2.8 that $k + l$ is even.

Remarks

1) In step (1) of the preceding proof k and l were chosen in an arbitrary manner, so we know by the Rule of Universal Generalization that the result obtained is true for all odd integers.
2) Although we may not realize it, we are using the Rule of Universal Specification (twice) in step (1). The first argument implicit in this step reads as follows.
   i) If n is an odd integer, then $n = 2r + 1$ for some integer r.
   ii) The integer k is a specific (but arbitrarily chosen) odd integer.
   iii) Therefore we may write $k = 2a + 1$ for some (specific) integer a.
3) In step (1) we do not have $k = 2a + 1$ and $l = 2a + 1$. Since k, l are arbitrarily chosen, it *may* be the case that $k = l$ — and when this happens we have $2a + 1 = k = l = 2b + 1$, from which it follows that $a = b$. [Since k may not equal l, it follows

that $(k-1)/2 = a$ may not equal $b = (l-1)/2$. Thus we should use the different variables a and b.]

Before we proceed with another theorem — written in the more conventional manner — let us examine the following.

EXAMPLE 2.57

Consider the following statement for the universe of integers.

If n is an integer, then $n^2 = n$ — or, $\forall n\ [n^2 = n]$.

Now for $n = 0$ it is true that $n^2 = 0^2 = 0 = n$. And if $n = 1$, it is also true that $n^2 = 1^2 = 1 = n$. However, we *cannot* conclude $n^2 = n$ for every integer n. The Rule of Universal Generalization does *not* apply here, for we cannot consider the choice of 0 (or 1) as an arbitrarily chosen integer. If $n = 2$, we have $n^2 = 4 \neq 2 = n$, and this one counterexample is enough to tell us that the given statement is false. However, either replacement — namely, $n = 0$ or $n - 1$ — is enough to establish the truth of the statement:

For some integer n, $n^2 = n$ — or, $\exists n\ [n^2 = n]$.

We close — at last — with three results to demonstrate how we shall write proofs throughout the remainder of the text.

THEOREM 2.3

For all integers k and l, if k and l are both odd, then their product kl is also odd.

Proof: Since k and l are both odd, we may write $k = 2a + 1$ and $l = 2b + 1$, for some integers a and b — because of Definition 2.8. Then the product $kl = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$, where $2ab + a + b$ is an integer. Therefore, by Definition 2.8 once again, it follows that kl is odd.

The preceding proof is an example of a direct proof. In our next example we shall prove a result in three ways: first by a direct argument (or proof), then by the contrapositive method, and finally by the method of proof by contradiction. [For the (method of) proof by contradiction we put in some extra details, since this is our first opportunity to use this technique.] The reader should not assume, however, that every theorem can be so readily proved in a variety of ways.

THEOREM 2.4

If m is an even integer, then $m + 7$ is odd.

Proof:

1) Since m is even, we have $m = 2a$ for some integer a. Then $m + 7 = 2a + 7 = 2a + 6 + 1 = 2(a + 3) + 1$. Since $a + 3$ is an integer, we know that $m + 7$ is odd.
2) Suppose that $m + 7$ is not odd, hence even. Then $m + 7 = 2b$ for some integer b and $m = 2b - 7 = 2b - 8 + 1 = 2(b - 4) + 1$, where $b - 4$ is an integer. Hence m is odd. [The result follows because the statements $\forall m\ [p(m) \rightarrow q(m)]$ and $\forall m[\neg q(m) \rightarrow \neg p(m)]$ are logically equivalent.]

3) Now assume that m is even and that $m + 7$ is also even. (This assumption is the negation of what we want to prove.) Then $m + 7$ even implies that $m + 7 = 2c$ for some integer c. And, consequently, $m = 2c - 7 = 2c - 8 + 1 = 2(c - 4) + 1$ with $c - 4$ an integer, so m is odd. Now we have our contradiction. We started with m even and deduced m odd — an impossible situation, since no integer can be both even and odd. How did we arrive at this dilemma? Simple — we made a mistake! This mistake is the false assumption — namely, $m + 7$ is even — that we wanted to believe at the start of the proof. Since the assumption is false, its negation is true, and so we now have $m + 7$ odd.

The second and third proofs for Theorem 2.4 appear to be somewhat similar. This is because the contradiction we derived in the third proof arises from the hypothesis of the theorem and its negation. We shall see as we progress (as early as the next chapter) that a contradiction may also be obtained by deriving the negation of a known fact — a fact that is *not* the hypothesis of the theorem we are attempting to prove. For now, however, let us think about this similarity a little more. Suppose we start with the open statements $p(m)$ and $q(m)$ — for a prescribed universe — and consider a theorem of the form $\forall m\ [p(m) \rightarrow q(m)]$. If we try to prove this result by the contrapositive method, then we shall actually prove the logically equivalent statement $\forall m\ [\neg q(m) \rightarrow \neg p(m)]$. To do so we assume the truth of $\neg q(m)$ (for any specific but arbitrarily chosen m in the universe) and show that this leads to the truth of $\neg p(m)$. On the other hand, if we wish to prove the theorem $\forall m\ [p(m) \rightarrow q(m)]$ by the method of proof by contradiction, then we assume that the statement $\forall m\ [p(m) \rightarrow q(m)]$ is false. This amounts to the fact that $p(m) \rightarrow q(m)$ is false for at least one replacement for m from the universe — that is, there is some element m in the universe for which $p(m)$ is true and $q(m)$ is false [or $\neg q(m)$ is true]. We then use the truth of $p(m)$ and $\neg q(m)$ to derive a contradiction. [In the third proof of Theorem 2.4 we obtained $p(m) \wedge \neg p(m)$.] These two methods can be compared symbolically in the following — where m is specific but arbitrarily chosen for the method of contraposition.

	Assumption	**Result Derived**
Contraposition	$\neg q(m)$	$\neg p(m)$
Contradiction	$p(m)$ and $\neg q(m)$	F_0

In general, when we are able to establish a theorem by either a direct proof or an indirect proof, the direct approach is less cumbersome than an indirect approach. (This certainly appears to be the case for the three proofs presented for Theorem 2.4.) When we do not have any prescribed directions given for attempting the proof of a certain theorem, we might start with a direct approach. If we succeed, then all is well. If not, then we might consider trying to find a counterexample to what we thought was a theorem. Should our search for a counterexample fail, then we might consider an indirect approach. We might prove the contrapositive of the theorem, or obtain a contradiction, as we did in the third proof of Theorem 2.4, by assuming the truth of the hypothesis and the truth of the negation of the conclusion (for some element m in the universe) in the given theorem.

We close this section with one more indirect proof by the method of contraposition.

THEOREM 2.5 For all positive real numbers x and y, if the product xy exceeds 25, then $x > 5$ or $y > 5$.

Proof: Consider the negation of the conclusion — that is, suppose that $0 < x \leq 5$ and $0 < y \leq 5$. Under these circumstances we find that $0 = 0 \cdot 0 < x \cdot y \leq 5 \cdot 5 = 25$, so the product

xy does *not* exceed 25. (This indirect method of proof now establishes the given statement, since we know that an implication is logically equivalent to its contrapositive.)

EXERCISES 2.5

1. In Example 2.52 why did we stop at 26 and not at 28?

2. In Example 2.52 why didn't we include the odd integers between 2 and 26?

3. Use the method of exhaustion to show that every even integer between 30 and 58 (including 30 and 58) can be written as a sum of at most three perfect squares.

4. Let n be a positive integer greater than 1. We call n *prime* if the only positive integers that (exactly) divide n are 1 and n itself. For example, the first seven primes are 2, 3, 5, 7, 11, 13, and 17. (We shall learn more about primes in Chapter 4.) Use the method of exhaustion to show that every integer in the universe 4, 6, 8, ..., 36, 38 can be written as the sum of two primes.

5. For each of the following (universes and) pairs of statements, use the Rule of Universal Specification, in conjunction with Modus Ponens and Modus Tollens, in order to fill in the blank line so that a valid argument results.

a) [The universe comprises all real numbers.]
All integers are rational numbers.
The real number π is not a rational number.
$\therefore$ ________________

b) [The universe comprises the present population of the United States.]
All librarians know the Library of Congress Classification System.
____
$\therefore$ Margaret knows the Library of Congress Classification System.

c) [The same universe as in part (b).]
____
Sondra is an administrative director.
$\therefore$ Sondra knows how to delegate authority.

d) [The universe consists of all quadrilaterals in the plane.]
All rectangles are equiangular.
____
$\therefore$ Quadrilateral $MNPQ$ is not a rectangle.

6. Determine which of the following arguments are valid and which are invalid. Provide an explanation for each answer. (Let the universe consist of all people presently residing in the United States.)

a) All mail carriers carry a can of mace.
Mrs. Bacon is a mail carrier.
Therefore Mrs. Bacon carries a can of mace.

b) All law-abiding citizens pay their taxes.
Mr. Pelosi pays his taxes.
Therefore Mr. Pelosi is a law-abiding citizen.

c) All people who are concerned about the environment recycle their plastic containers.
Margarita is not concerned about the environment.
Therefore Margarita does not recycle her plastic containers.

7. For a prescribed universe and any open statements $p(x)$, $q(x)$ in the variable x, prove that

a) $\exists x\, [p(x) \vee q(x)] \Longleftrightarrow \exists x\, p(x) \vee \exists x\, q(x)$

b) $\forall x\, [p(x) \wedge q(x)] \Longleftrightarrow \forall x\, p(x) \wedge \forall x\, q(x)$

8. a) Let $p(x), q(x)$ be open statements in the variable x, with a given universe. Prove that

$$\forall x\, p(x) \vee \forall x\, q(x) \Rightarrow \forall x\, [p(x) \vee q(x)].$$

[That is, prove that when the statement $\forall x\, p(x) \vee \forall x\, q(x)$ is true, then the statement $\forall x\, [p(x) \vee q(x)]$ is true.]

b) Find a counterexample for the converse in part (a). That is, find open statements $p(x), q(x)$ and a universe such that $\forall x\, [p(x) \vee q(x)]$ is true, while $\forall x\, p(x) \vee \forall x\, q(x)$ is false.

9. Provide the reasons for the steps verifying the following argument. (Here a denotes a specific but arbitrarily chosen element from the given universe.)

$$\begin{array}{l} \forall x\, [p(x) \rightarrow (q(x) \wedge r(x))] \\ \underline{\forall x\, [p(x) \wedge s(x)]} \\ \therefore \forall x\, [r(x) \wedge s(x)] \end{array}$$

Steps	**Reasons**
1) $\forall x\, [p(x) \rightarrow (q(x) \wedge r(x))]$	
2) $\forall x\, [p(x) \wedge s(x)]$	
3) $p(a) \rightarrow (q(a) \wedge r(a))$	
4) $p(a) \wedge s(a)$	
5) $p(a)$	
6) $q(a) \wedge r(a)$	
7) $r(a)$	
8) $s(a)$	
9) $r(a) \wedge s(a)$	
10) $\therefore \forall x\, [r(x) \wedge s(x)]$	

10. Provide the missing reasons for the steps verifying the following argument:

$$\begin{array}{l} \forall x\, [p(x) \vee q(x)] \\ \exists x\, \neg p(x) \\ \forall x\, [\neg q(x) \vee r(x)] \\ \underline{\forall x\, [s(x) \rightarrow \neg r(x)]} \\ \therefore \exists x\, \neg s(x) \end{array}$$

Steps	Reasons
1) $\forall x\,[p(x) \vee q(x)]$	Premise
2) $\exists x\, \neg p(x)$	Premise
3) $\neg p(a)$	Step (2) and the definition of the truth for $\exists x\, \neg p(x)$. [Here a is an element (replacement) from the universe for which $\neg p(x)$ is true.] The reason for this step is also referred to as the *Rule of Existential Specification.*
4) $p(a) \vee q(a)$	
5) $q(a)$	
6) $\forall x\,[\neg q(x) \vee r(x)]$	
7) $\neg q(a) \vee r(a)$	
8) $q(a) \rightarrow r(a)$	
9) $r(a)$	
10) $\forall x\,[s(x) \rightarrow \neg r(x)]$	
11) $s(a) \rightarrow \neg r(a)$	
12) $r(a) \rightarrow \neg s(a)$	
13) $\neg s(a)$	
14) $\therefore \exists x\, \neg s(x)$	Step (13) and the definition of the truth for $\exists x\, \neg s(x)$. The reason for this step is also referred to as the *Rule of Existential Generalization.*

11. Write the following argument in symbolic form. Then either verify the validity of the argument or explain why it is invalid. [Assume here that the universe comprises all adults (18 or over) who are presently residing in the city of Las Cruces (in New Mexico). Two of these individuals are Roxe and Imogene.]

All credit union employees must know COBOL. All credit union employees who write loan applications must know Excel.[†] Roxe works for the credit union, but she doesn't know Excel. Imogene knows Excel but doesn't know COBOL. Therefore Roxe doesn't write loan applications and Imogene doesn't work for the credit union.

12. Give a direct proof (as in Theorem 2.3) for each of the following.

a) For all integers k and l, if k, l are both even, then $k + l$ is even.

b) For all integers k and l, if k, l are both even, then kl is even.

13. For each of the following statements provide an indirect proof [as in part (2) of Theorem 2.4] by stating and proving the contrapositive of the given statement.

a) For all integers k and l, if kl is odd, then k, l are both odd.

b) For all integers k and l, if $k + l$ is even, then k and l are both even or both odd.

14. Prove that for every integer n, if n is odd, then n^2 is odd.

15. Provide a proof by contradiction for the following: For every integer n, if n^2 is odd, then n is odd.

16. Prove that for every integer n, n^2 is even if and only if n is even.

17. Prove the following result in three ways (as in Theorem 2.4): If n is an odd integer, then $n + 11$ is even.

18. Let m, n be two positive integers. Prove that if m, n are perfect squares, then the product mn is also a perfect square.

19. Prove or disprove: If m, n are positive integers and m, n are perfect squares, then $m + n$ is a perfect square.

20. Prove or disprove: There exist positive integers m, n, where m, n, and $m + n$ are all perfect squares.

21. Prove that for all real numbers x and y, if $x + y \geq 100$, then $x \geq 50$ or $y \geq 50$.

22. Prove that for every integer n, $4n + 7$ is odd.

23. Let n be an integer. Prove that n is odd if and only if $7n + 8$ is odd.

24. Let n be an integer. Prove that n is even if and only if $31n + 12$ is even.

2.6 Summary and Historical Review

This second chapter has introduced some of the fundamentals of logic — in particular, some of the rules of inference and methods of proof necessary for establishing mathematical theorems.

The first systematic study of logical reasoning is found in the work of the Greek philosopher Aristotle (384–322 B.C.). In his treatises on logic Aristotle presented a collection of principles for deductive reasoning. These principles were designed to provide a foundation

[†]The Excel spreadsheet is a product of Microsoft, Inc.

for the study of all branches of knowledge. In a modified form, this type of logic was taught up to and throughout the Middle Ages.

Aristotle (384–322 B.C.)

The German mathematician Gottfried Wilhelm Leibniz (1646–1716) is often considered the first scholar who seriously pursued the development of symbolic logic as a universal scientific language. This he professed in his essay *De Arte Combinatoria*, published in 1666. His research in the area of symbolic logic, carried out from 1679 to 1690, gave considerable impetus to the creation of this mathematical discipline.

Following the work by Leibniz, little change took place until the nineteenth century, when the English mathematician George Boole (1815–1864) created a system of mathematical logic that he introduced in 1847 in the pamphlet *The Mathematical Analysis of Logic, Being an Essay Towards a Calculus of Deductive Reasoning*. In the same year, Boole's countryman Augustus DeMorgan (1806–1871) published *Formal Logic; or, the Calculus of Inference, Necessary and Probable*. In some ways this treatise extended Boole's work

George Boole (1815–1864)

considerably. Then, in 1854, Boole detailed his ideas and further research in the notable work *An Investigation in the Laws of Thought, on Which Are Founded the Mathematical Theories of Logic and Probability*. The American logician Charles Sanders Peirce (1839–1914), who was also an engineer and philosopher, introduced the formal concept of the *quantifier* into the study of symbolic logic.

The concepts formulated by Boole were thoroughly examined in the work of another German scholar, Ernst Schröder (1841–1902). These results are known collectively as *Vorlesungen über die Algebra der Logik*; they were published in the period from 1890 to 1895.

Further developments in the area saw an even more modern approach evolve in the work of the German logician Gottlieb Frege (1848–1925) between 1879 and 1903. This work significantly influenced the monumental *Principia Mathematica* (1910–1913) by England's Alfred North Whitehead (1861–1947) and Bertrand Russell (1872–1970). Here what was begun by Boole was finally brought to fruition. Thanks to this remarkable effort and the work of other twentieth-century mathematicians and logicians, in particular the comprehensive *Grundlagen der Mathematik* (1934–1939) of David Hilbert (1862–1943) and Paul Bernays (1888–1977), the more polished techniques of contemporary mathematical logic are now available.

Several sections of this chapter stressed the importance of proof. In mathematics a proof bestows authority on what might otherwise be dismissed as mere opinion. Proof embodies the power and majesty of pure reason. But even more than that, it suggests new mathematical ideas. Our concept of proof goes hand in hand with the notion of a *theorem* — a mathematical statement the truth of which has been confirmed by means of a logical argument, namely, a *proof*. For those who feel they can ignore the importance of logic and the rules of inference, we submit the following words of wisdom spoken by Achilles in Lewis Carroll's *What the Tortoise Said to Achilles*: "Then Logic would take you by the throat, and force you to do it!"

Comparable coverage of the material presented in this chapter can be found in Chapters 2 and 11 of the text by K. A. Ross and C. R. B. Wright [11]. The first two chapters of the text by S. S. Epp [3] provide many examples and some computer science applications for those who wish to see more on logic and proof at a very readable introductory level. The text by H. Delong [2] provides an historical survey of mathematical logic, together with an examination of the nature of its results and the philosophical consequences of these results. This is also the case with the texts by H. Eves and C. V. Newsom [4], R. R. Stoll [13], and R. L. Wilder [14], wherein the relationships among logic, proof, and set theory (the topic of our next chapter) are examined in their roles in the foundations of mathematics.

For more on resolution (introduced in Exercise 13 of Section 2.3) and automated reasoning, the reader should examine the texts by J. H. Gallier [6] and M. R. Genesereth and N. J. Nilsson [7].

The text by E. Mendelson [9] provides an interesting intermediate introduction for those readers who wish to pursue additional topics in mathematical logic. A somewhat more advanced treatment is given in the work of S. C. Kleene [8]. Accounts of other work in mathematical logic are presented in the compendium edited by J. Barwise [1].

The objective of the works by D. Fendel and D. Resek [5] and R. P. Morash [10] is to prepare the student with a calculus background for the more theoretical mathematics found in abstract algebra and real analysis. Each of these texts provides an excellent introduction to the basic methods of proof. The unique text by D. Solow [12] is devoted entirely to introducing the reader who has a background in high school mathematics to the primary techniques used in writing mathematical proofs.

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SUPPLEMENTARY EXERCISES

1. Construct the truth table for

$$p \leftrightarrow [(q \wedge r) \rightarrow \neg(s \vee r)].$$

2. a) Construct the truth table for

$$(p \rightarrow q) \wedge (\neg p \rightarrow r).$$

b) Translate the statement in part (a) into words such that the word "not" does not appear in the translation.

3. Let p, q, and r denote primitive statements. Prove or disprove (provide a counterexample for) each of the following.

a) $[p \leftrightarrow (q \leftrightarrow r)] \Longleftrightarrow [(p \leftrightarrow q) \leftrightarrow r]$

b) $[p \rightarrow (q \rightarrow r)] \Longleftrightarrow [(p \rightarrow q) \rightarrow r]$

4. Express the negation of the statement $p \leftrightarrow q$ in terms of the connectives $\wedge$ and $\vee$.

5. Write the following statement as an implication in two ways, each in the *if-then* form: Either Kaylyn practices her piano lessons or she will not go to the movies.

6. Let p, q, r denote primitive statements. Write the converse, inverse, and contrapositive of

a) $p \rightarrow (q \wedge r)$ **b)** $(p \vee q) \rightarrow r$

7. a) For primitive statements p, q, find the dual of the statement $(\neg p \wedge \neg q) \vee (T_0 \wedge p) \vee p$.

b) Use the laws of logic to show that your result from part (a) is logically equivalent to $p \wedge \neg q$.

8. Let p, q, r, and s be primitive statements. Write the dual of each of the following compound statements.

a) $(p \vee \neg q) \wedge (\neg r \vee s)$

b) $p \rightarrow (q \wedge \neg r \wedge s)$

c) $[(p \vee T_0) \wedge (q \vee F_0)] \vee [r \wedge s \wedge T_0]$

9. For each of the following, fill in the blank with the word *converse*, *inverse*, or *contrapositive* so that the result is a true statement.

a) The converse of the inverse of $p \rightarrow q$ is the ______________ of $p \rightarrow q$.

b) The converse of the inverse of $p \rightarrow q$ is the ______________ of $q \rightarrow p$.

c) The inverse of the converse of $p \rightarrow q$ is the ______________ of $p \rightarrow q$.

d) The inverse of the converse of $p \rightarrow q$ is the ______________ of $q \rightarrow p$.

e) The inverse of the contrapositive of $p \rightarrow q$ is the ______________ of $p \rightarrow q$.

10. Establish the validity of the argument

$$[(p \rightarrow q) \wedge [(q \wedge r) \rightarrow s] \wedge r] \rightarrow (p \rightarrow s).$$

11. Prove or disprove each of the following, where p, q, and r are any statements.

a) $[(p \veebar q) \veebar r] \Longleftrightarrow [p \veebar (q \veebar r)]$

b) $[p \veebar (q \rightarrow r)] \Longleftrightarrow [(p \veebar q) \rightarrow (p \veebar r)]$

12. Write the following argument in symbolic form. Then either establish the validity of the argument or provide a counterexample to show that it is invalid.

> If it is cool this Friday, then Craig will wear his suede jacket if the pockets are mended. The forecast for Friday calls for cool weather, but the pockets have not been mended. Therefore Craig won't be wearing his suede jacket this Friday.

13. Consider the open statement

$$p(x, y): \quad y - x = y + x^2$$

where the universe for each of the variables x, y comprises all integers. Determine the truth value for each of the following statements.

a) $p(0, 0)$ **b)** $p(1, 1)$

c) $p(0, 1)$ **d)** $\forall y\ p(0, y)$

e) $\exists y\ p(1, y)$ **f)** $\forall x\ \exists y\ p(x, y)$

g) $\exists y\ \forall x\ p(x, y)$ **h)** $\forall y\ \exists x\ p(x, y)$

14. Determine whether each of the following statements is true or false. If false, provide a counterexample. The universe comprises all integers.

a) $\forall x\ \exists y\ \exists z\ (x = 7y + 5z)$

b) $\forall x\ \exists y\ \exists z\ (x = 4y + 6z)$

15. Suppose two opposite corner squares are removed from an 8×8 chessboard — as in part (a) of Fig. 2.4. Can the remaining 62 squares be covered by 31 dominos (rectangles consisting of two adjacent squares — one white and the other blue, as shown in the figure)? (When a domino is placed on the chessboard, a square of a given color need not be placed on a square of the same color.)

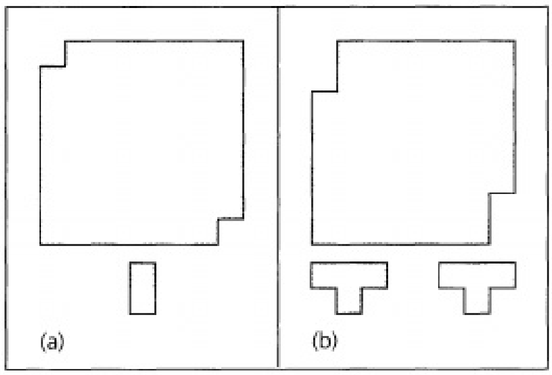

Figure 2.4

16. In part (b) of Fig. 2.4 we have an 8×8 chessboard where two squares (one blue and one white) have been removed from each of two opposite corners. Can the remaining 60 squares be covered by 15 T-shaped figures (of three white squares and one blue one, or three blue squares and one white one — as shown in the figure)? [The reader may wish to verify that a 4×4 chessboard (of all 16 squares) can be covered by four of the T-shaped figures. Then it follows that an 8×8 chessboard (of all 64 squares) can be covered by 16 of the T-shaped figures.]