Module - 2 Relations and Functions

Cartesian Product of sets :

Let A and B be 2 sets. Then the set of all ordered pairs (a,b), where $a \in A$ and $b \in B$, is called Cartesian Product (or) Cross Product (or) Product set of A and B and is denoted by $A \times B$.

Thus $A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$.

Note : 1) $A \times B$ is not same as $B \times A$, because

$B \times A = \{(b,a) \mid b \in B \text{ and } a \in A\}$.

$\therefore (a,b) \neq (b,a)$ in general.

2) If (a,b) and (c,d) are ordered pairs, then $(a,b) = (c,d)$ if and only if $a = c$, $b = d$.

3) If A and B are finite sets with $|A| = m$, $|B| = n$, then

$|A \times B| = mn = |A||B|$.

4) $|P(A \times B)| = 2^{mn}$ ie $A \times B$ has 2^{mn} subsets where $|A| = m$, $|B| = n$. for ex: if A has 5 elements, B has 6 elements then $A \times B$ has $2^{5\times6} = 2^{30}$ subsets.

Note : Power set : If a set A has n elements, then the power set of A has 2^n elements.

Problems :

1) Let $A = \{1,2,3,4\}$ $B = \{3,4,5,6\}$ $C = \{2,4,6\}$. find $A \times B$, $B \times A$, $A \times (B \cup C)$, $(A \cap B) \times C$, $(A \times B) \cap (B \times C)$, $(A \times B) - (B \times C)$.

Soln :- $A \times B = \{(1,3)(1,4)(1,5)(1,6)(2,3)(2,4)(2,5)(2,6)(3,3)(3,4)(3,5)(3,6)(4,3)(4,4)(4,5)(4,6)\}$

$B \times A = \{(3,1)(3,2)(3,3)(3,4)(4,1)(4,2)(4,3)(4,4)(5,1)(5,2)(5,3)(5,4)(6,1)(6,2)(6,3)(6,4)\}$

$B \cup C = \{2,3,4,5,6\}$.

$A \times (B \cup C) = \{(1,2)(1,3)(1,4)(1,5)(1,5)(2,2)(2,3)(2,4)(2,5)(2,6)(3,2)(3,3)(3,4)(3,5)(3,6)(4,2)(4,3)(4,4)(4,5)(4,6)\}$

$(A\cap B) = \{3,4\}$, $C = \{2,4,6\}$.

$(A\cap B)\times C = \{(3,2)\ (3,4)\ (3,6)\ (4,2)\ (4,4)\ (4,6)\}$.

$(B\times C) = \{(3,2)\ (3,4)\ (3,6)\ (4,2)\ (4,4)\ (4,6)\ (5,2)\ (5,4)\ (5,6)\ (6,2)$
$(6,4)\ (6,6)\}$.

$\therefore (A\times B)\cap(B\times C) = \{(3,4)\ (3,6)\ (4,4)\ (4,6)\}$.

$(A\times B)-(B\times C) = \{(1,3)\ (1,4)\ (1,5)\ (1,6)\ (2,3)\ (2,4)\ (2,5)\ (2,6)\ (3,3)$
$(3,5)\ (4,3)\ (4,5)\}$.

elements in $(A\times B)$
but not in $(B\times C)$.

2) For any non-empty sets A, B, C, prove the following results:

i) $A\times(B\cup C) = (A\times B)\cup(A\times C)$

ii) $(A\cup B)\times C = (A\times C)\cup(B\times C)$

iii) $A\times(B\cap C) = (A\times B)\cap(A\times C)$.

iv) $(A\cap B)\times C = (A\times C)\cap(B\times C)$

v) $A\times(B-C) = (A\times B)-(A\times C)$.

Soln:- For any ordered pair (x,y), we have

i) $(x,y)\in\{A\times(B\cup C)\} \Leftrightarrow x\in A$ and $y\in B\cup C$

$\Leftrightarrow x\in A$ and, $(y\in B$ or $y\in C)$.

$\Leftrightarrow x\in A$ and $y\in B$, or $x\in A$ and $y\in C$.

$\Leftrightarrow (x,y)\in A\times B$ or $(x,y)\in A\times C$

$\Leftrightarrow (x,y)\in\{(A\times B)\cup(A\times C)\}$.

Thus $A\times(B\cup C) = (A\times B)\cup(A\times C)$.

ii) $(x,y)\in\{(A\cup B)\times C\} \Leftrightarrow x\in A\cup B$, and $y\in C$.

$\Leftrightarrow (x\in A$ or $x\in B)$ and $y\in C$

$\Leftrightarrow x\in A$ and $y\in C$ or $x\in B$ and $y\in C$.

$\Leftrightarrow (x,y)\in A\times C$ or $(x,y)\in B\times C$.

$\Leftrightarrow (x,y)\in\{(A\times C)\cup(B\times C)\}$.

Thus $(A\cup B)\times C = (A\times C)\cup(B\times C)$.

iii) $(x,y) \in \{A \times (B \cap C)\} \Leftrightarrow x \in A$ and $y \in B \cap C$.

$\Leftrightarrow x \in A$ and $(y \in B$ and $y \in C)$.

$\Leftrightarrow x \in A$ and $y \in B$, and $x \in A$ and $y \in C$.

$\Leftrightarrow (x,y) \in A \times B$ and $(x,y) \in A \times C$.

$\Leftrightarrow (x,y) \in \{(A \times B) \cap (A \times C)\}$

Thus $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

iv) $(x,y) \in (A \cap B) \times C \Leftrightarrow x \in A \cap B$ and, $y \in C$.

$\Leftrightarrow (x \in A$ and $x \in B)$, and $y \in C$.

$\Leftrightarrow x \in A$ and $y \in C$ and $x \in B$ and $y \in C$

$\Leftrightarrow (x,y) \in (A \times C)$ and $(x,y) \in (B \times C)$

$\Leftrightarrow (x,y) \in \{(A \times C) \cap (B \times C)\}$.

Thus $(A \cap B) \times C = (A \times C) \cap (B \times C)$.

v) $A \times (B - C) = (A \times B) - (A \times C)$.

$(x,y) \in \{A \times (B-C)\} \Leftrightarrow x \in A$ and $y \in B - C$.

$\Leftrightarrow x \in A$ and $(y \in B$, and $y \notin C)$

$\Leftrightarrow (x \in A$ and $y \in B)$, and $(x \in A$ and $y \notin C)$

$\Leftrightarrow (x,y) \in (A \times B)$ and $(x,y) \notin (A \times C)$

$\Leftrightarrow (x,y) \in \{(A \times B) - (A \times C)\}$.

Thus $A \times (B - C) = (A \times B) - (A \times C)$.

3) suppose $A, B, C \subseteq Z \times Z$ with $A = \{(x,y) \mid y = 5x - 1\}$,

QP $B = \{(x,y) \mid y = 6x\}$, $C = \{(x,y) \mid 3x - y = -7\}$.

find i) $A \cap B$ ii) $B \cap C$ iii) $\overline{A} \cup \overline{C}$ iv) $\overline{B} \cup \overline{C}$.

soln:- i) $(x,y) \in A \cap B \Leftrightarrow (x,y) \in A$ and $(x,y) \in B$.

$\Leftrightarrow y = 5x - 1$ and $y = 6x$.

$\Leftrightarrow 5x - 1 = y = 6x$.

$\Leftrightarrow x = -1, y = -6$.

$\therefore A \cap B = \{(-1,-6)\}$.

ii) $(x,y) \in B \cap C \iff (x,y) \in B$ and $(x,y) \in C$.

$\iff y = 6x$ and $3x - y = -7$.

$\iff y = 6x$ and $y = 3x + 7$

$\iff 6x = y = 3x + 7$

$\iff 3x = 7$ i.e $x = 7/3$

which is not possible, because $x \in Z$.

Thus $(B \cap C) = \phi$.

iii) we have $\bar{A} \cup \bar{C} = \overline{A \cap C}$

$\Rightarrow \overline{\bar{A} \cup \bar{C}} = \overline{\overline{A \cap C}} = A \cap C$.

$(x,y) \in A \cap C \iff (x,y) \in A$ and $(x,y) \in C$.

$\iff y = 5x - 1$ and $3x - y = -7$.

$\iff y = 5x - 1$ and $y = 3x + 7$

$\iff 5x - 1 = y = 3x + 7$

$\iff x = 4,\ y = 19$.

Thus $\overline{\bar{A} \cup \bar{C}} = A \cap C = \{(4, 19)\}$.

iv) we have $\bar{B} \cup \bar{C} = \overline{B \cap C}$.

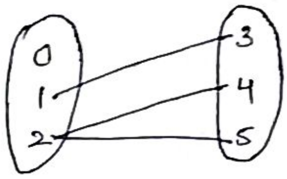
from (ii), $B \cap C = \phi$.

$\therefore \overline{B \cap C} = \bar{B} \cup \bar{C} = Z \times Z$ (universal set).

<u>Relations</u> :- Let A and B be 2 sets. Then a subset of $A \times B$ is called a binary relation or just a relation from A to B. Thus (if R is a relation from A to B, then R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$), and conversely if R is a set of ordered pairs (a,b) where $a \in A$ and $b \in B$, then R is a relation from A to B.

$\Rightarrow$ need not be all

If $(a,b) \in R$, we say that "a is related to b by R". this is denoted by aRb. If R is a relation from A to A i.e. R is a subset of $A \times A$, we say that R is a binary relation on A.



(a)

<u>Ex</u> :- $A = \{0,1,2\}$ and $B = \{3,4,5\}$

Let $R = \{(1,3)\ (2,4)\ (2,5)\}$

R is a subset of $A \times B$. This is represented by the diagram (a) and this diagram is called the arrow diagram.

<u>Note</u> :- If A has 'm' elements and B has 'n' elements, then the no. of relations from A to B is 2^{mn}.

<u>Problems</u> :-

1) Let $A = \{1,2,3,4,6\}$ and R be the relation on A defined by $(a,b) \in R$ if and only if a is a multiple of b. write down R as a set of ordered pairs.

<u>soln</u>:- $R = \{(a,b) \mid a, b \in A$ and a is a multiple of $b\}$

$R = \{(1,1)\ (2,1)\ (2,2)\ (3,1)\ (3,3)\ (4,1)\ (4,2)\ (4,4)\ (6,1)\ (6,2)\ (6,3)\ (6,6)\}$

2) Let A and B be finite sets with $|B| = 3$. If there are 4096 relations from A to B, what is $|A| = ?$

<u>soln</u>:- If $|A| = m$, $|B| = n$, then there are 2^{mn} relations from A to B.

Given $|B| = n = 3$. and $2^{mn} = 4096$.

$\therefore \quad 2^{3m} = 4096$

$\Rightarrow \quad 3m \log_e 2 = \log_e 4096$

$$\Rightarrow \quad m = \frac{\log_e 4096}{3 \times \log_e 2} = 4$$

Thus $|A| = 4$.

8) Let $A = \{1,2,3\}$ and $B = \{2,4,5\}$. Determine the following:

Q) i) $|A \times B|$.

ii) No. of relations from A to B.

iii) No. of binary relations on A.

iv) No. of relations from A to B that contain $(1,2)$ & $(1,5)$.

v) No. of relations from A, B that contain exactly 5 ordered pair

vi) No. of binary relations on A that contain atleast 7 ordered pairs

soln:- Given $|A| = m = 3$, $|B| = n = 3$.

i) $|A \times B| = mn = 9$.

ii) No. of relations from A to B is $2^{mn} = 2^9 = 512$.

iii) No. of binary relations on A is $2^{mm} = 2^{m^2} = 2^9 = 512$.

iv) Let $R_1 = \{(1,2)(1,5)\}$. Every relation from A to B that contains the elements $(1,2)$ and $(1,5)$ is of the form $R_1 \cup R_2$, where R_2 is a subset of $\overline{R_1}$ in $A \times B$.

$\therefore$ No. of such relations $=$ no. of subsets of $\overline{R_1}$.

$= 2^7 \quad (\because |\overline{R_1}| = |A \times B| - |R_1| = 9 - 2 = 7)$

$= 128$

Thus there are 128 no. of relations from A to B that contain the elements $(1,2)$ and $(1,5)$.

v) Since $A \times B$ contains 9 ordered pairs, the no. of relations from A to B that contain exactly 5 ordered pairs = no. of ways of choosing 5 ordered pairs from 9 ordered pairs.

This no. is ${}^9C_5 = 126$.

vi) III^{ly}, the no. of binary relations on A that contain atleast 7 elements (ordered pairs) is ${}^9C_7 + {}^9C_8 + {}^9C_9 = 46$.

Matrix of a relation :- Consider the sets $A = \{a_1, a_2 \ldots a_m\}$, $B = \{b_1, b_2 \ldots b_n\}$ of orders m and n resp. Then $A \times B$ contains all ordered pairs of the form (a_i, b_j) $1 \le i \le m$, $1 \le j \le n$ which are mn in number.

Let R be a relation from A to B so that R is a subset of $A \times B$. Let $m_{ij} = (a_i, b_j)$ and

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases} \rightarrow \text{zero-one matrix.}$$

The $m \times n$ matrix formed by m_{ij} is called the relation matrix for R or the adjacency matrix or zero-one matrix for R and is denoted by M_R or $M(R)$.

Rows of M_R correspond to elements of A and columns correspond to elements of B.

when $B = A$, then M_R is an $n \times n$ matrix whose elements are $m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in R \\ 0 & \text{if } (a_i, a_j) \notin R. \end{cases}$

Ex :- 1) $A = \{1, 2, 3, 4\}$ $B = \{4, 5\}$ and $R = \{(1,4)\ (2,5)\}$

$$\therefore M_R = \begin{array}{c c} & \begin{array}{cc} 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

2) $A = \{p, q, r\}$ $R = \{(p,p)\ (p,q)\ (r,r)\ (r,q)\}$ then

$$M_R = \begin{array}{c c} & \begin{array}{ccc} p & q & r \end{array} \\ \begin{array}{c} p \\ q \\ r \end{array} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{array}$$

is defined only on $R: A \to A$, never on $R: A \to B$

Digraph of a relation :- Let R be a relation on a finite set A. To get the digraph of R, we follow the following procedure :-

1. Draw a small circle or bullet for each of the elements of A and label it with the corresponding element of A. These circles are called vertices or nodes.

2. Draw an arrow, called an edge from a vertex x to a vertex y if and only if $(x,y) \in R$.

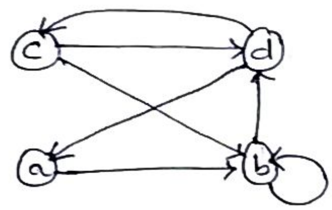
Note :- 1) In a digraph, a vertex from which an edge leaves is called the origin or the <u>source</u> for that edge and a vertex where an edge ends is called the <u>terminus</u> for that edge.

2) A vertex which is neither a source nor a terminus of an edge is known as an <u>isolated vertex</u>.

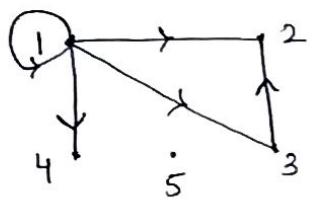
3) An edge for which the source and the terminus are one and the same vertex is called a <u>loop</u>.

4) The no. of edges (arrows) terminating at a vertex is called the <u>in-degree</u> of that vertex and the no. of edges (arrows) leaving a vertex is called the <u>out-degree</u> of that vertex.

<u>Ex</u> :- 1) $A = \{a, b, c, d\}$ $R = \{(a,b)\ (b,b)\ (b,d)\ (c,b)\ (c,d)\ (d,a)\ (d,c)\}$



2) $A = \{1, 2, 3, 4, 5\}$ $R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (3,2)\}$



<u>Operations on Relations</u> :

1) <u>Union and Intersection of Relations</u> :

Given the relations R_1 and R_2 from a set A to a set B, the union of R_1 and R_2, denoted by $R_1 \cup R_2$, is defined as a relation from A to B with the property that $(a,b) \in R_1 \cup R_2$ iff $(a,b) \in R_1$ or $(a,b) \in R_2$.

The intersection of R_1 and R_2, denoted by $R_1 \cap R_2$, is defined as a relation from A to B with the property that $(a,b) \in R_1 \cap R_2$ iff $(a,b) \in R_1$ and $(a,b) \in R_2$.

Complement of a Relation :

Given a relation R from a set A to a set B, the complement of R, denoted by $\overline{R}$, is defined as a relation from A to B with the property that $(a,b) \in \overline{R}$ iff $(a,b) \notin R$.

Converse of a Relation :-

Given a relation R from a set A to a set B, the converse of R denoted by R^C, is defined as a relation from B to A with the property that $(a,b) \in R^C$ iff $(b,a) \in R$.

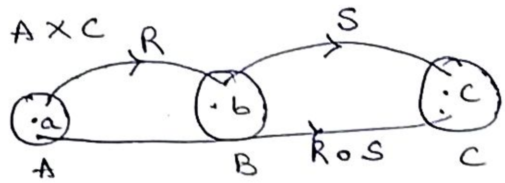
NOTE : 1) If M_R is the matrix of R, then $(M_R)^T$, the transpose of M_R, is the matrix of R^C.

2) $(R^C)^C = R$.

Composition of Relations :- consider a relation R from a set A to a set B and a relation S from the set B to the set C. Then the Product or the composition of R and S is a relation from the set A to the set C, denoted by RoS and is defined as if $a \in A$ and $c \in C$ then $(a,c) \in RoS$ iff there is some b in B such that $(a,b) \in R$ and $(b,c) \in S$

i.e. $RoS = \{(a,c) \mid a \in A,\ c \in C$ and $\exists\ b \in B$ with $(a,b) \in R$ and $(b,c) \in S\}$.

NOTE :- 1) $R \subseteq A \times B$, $S \subseteq B \times C$ & $RoS \subseteq A \times C$



2) $RoS \neq SoR$.

3) If R is a relation on A, then RoR is a relation on A, denoted by R^2 and $(RoR)oR$ is also a relation on A, denoted by R^3.

4) Let R be a relation from A to B and S be a relation from B to C, then the matrices of R, S and RoS satisfy $M(R).M(S) = M(RoS)$

5) $M(R^2) = [M(R)]^2$ and $M[R^n] = [M(R)]^n$, $n \in \mathbb{Z}^+$.

6) Let A, B, C, D be the sets and R, S, T be the relations from A to B, B to C and C to D resp, then $Ro(SoT) = (RoS)oT$.

Problems :-

1) Let $A = \{1, 2, 3, 4\}$ and let R be the relation on A defined by $x R y$ iff $y = 2x$.

i) write down R as a set of ordered pairs

ii) Draw the digraph of R.

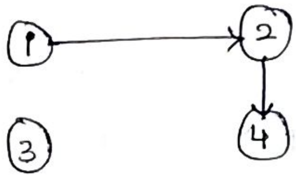
iii) Determine the in-degrees and out-degrees of the vertices in the digraph.

iv) write the matrix of R.

Soln :- i) for $x, y \in A$, $(x, y) \in R$ iff $y = 2x$.

$\therefore$ $R = \{(1, 2)\ (2, 4)\}$.

ii) Digraph of R is as shown below :



iii)

vertices	Indegree	out degree
1	0	1
2	1	1
3	0	0
4	1	0

iv) $$M(R) = M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

2) Let $A = \{1, 2, 3, 4, 6\}$ and R be the relation on A defined by $a R b$ iff a is a multiple of b.

i) Represent the relation R as a set of ordered pairs.

ii) Draw its digraph.

iii) Determine the indegrees and outdegrees of the vertices.

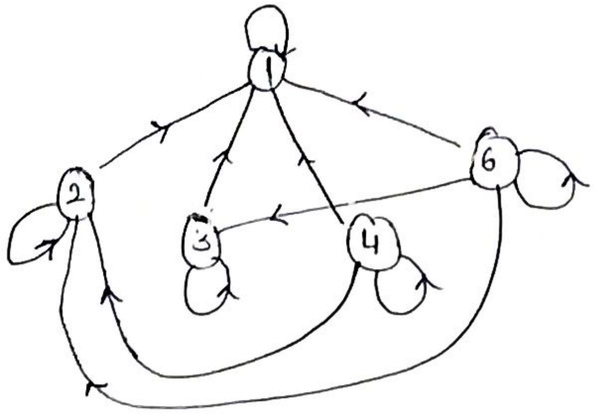
iv) write the matrix of R.

Soln :- Given $(a, b) \in R$ iff a is a multiple of b.

i) $R = \{(1,1)\ (2,1)\ (2,2)\ (3,1)\ (3,3)\ (4,1)\ (4,2)\ (4,4)\ (6,1)\ (6,2)$
$(6,3)\ (6,6)\}$.

iv) $M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}$

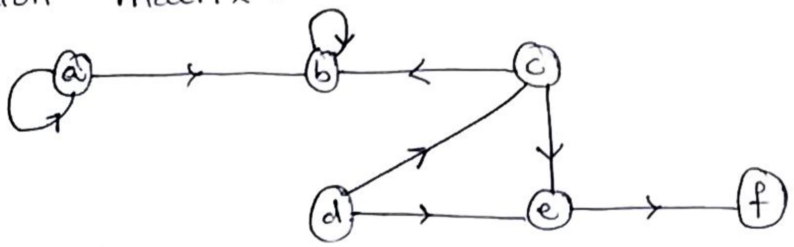
ii)



iii)

vertices	Indegree	out degree
1	5	1
2	3	2
3	2	2
4	1	3
6	1	4

3) For $A = \{a, b, c, d, e, f\}$ the digraph in the fig below rep--resents a relation on A. Determine R and the associated relation matrix.



<u>soln</u> :- $R = \{(a,a)\ (a,b)\ (b,b)\ (c,b)\ (c,e)\ (d,c)\ (d,e)\ (e,f)\}$.

$M(R) = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

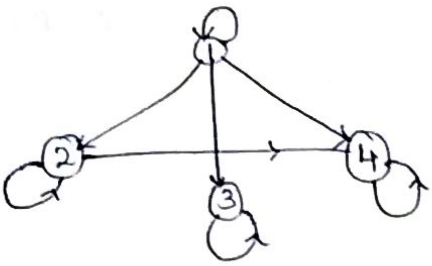
4) Let $A = \{1, 2, 3, 4\}$ and R be a relation on A defined by $(a,b) \in R$ iff $a \mid b$ (a divides b). write down R as a set of ordered pairs. write the matrix of this relation and draw the digraph of R. Also find find indegree and outdegree of all the vertices. R^2, R^3 and matrices of R^2 and R^3 and digraphs.

P.T.O.

Soln:- Given $A = \{1, 2, 3, 4\}$ and R is the relation on A defined by $(a, b) \in R$ iff $a | b$.

$\therefore \quad R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$$M(R) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Vertices	Indegree	Outdegree
1	1	4
2	2	2
3	2	1
4	3	1

$R^2 = R \circ R = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$\circ$

$\{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$R^2 = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,2)\ (2,4)\ (3,3)\ (4,4)\}$

$\therefore \boxed{R^2 = R}$

$R^3 = R^2 \circ R = R \circ R = R^2 = R.$

$\therefore \boxed{R^3 = R}$

Since R^2 and R^3 are same as R, matrices of R^2 and R^3 are same as matrix of R and digraphs of R^2 and R^3 are same of digraphs of R.

5) Let $A = \{1, 2, 3, 4\}$, $B = \{w, x, y, z\}$ and $C = \{5, 6, 7\}$. Also let R_1 be a relation from A to B; R_2 and R_3 be relations from B to C defined by

SP.

$R_1 = \{(1, x)\ (2, x)\ (3, y)\ (3, z)\}$

$R_2 = \{(w, 5)\ (x, 6)\}$ $\quad R_3 = \{(w, 5)\ (w, 6)\}$.

i) Find $R_1 \circ R_2$ and $R_1 \circ R_3$.

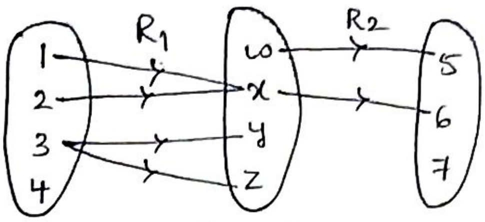
ii) Find $M(R_1)$, $M(R_2)$ and $M(R_1 \circ R_2)$

iii) Verify that $M(R_1 \circ R_2) = M(R_1) \cdot M(R_2)$

Soln :- i) $(1,x) \in R_1$ and $(x,6) \in R_2 \Rightarrow (1,6) \in R_1 \circ R_2$

$(2,x) \in R_1$ and $(x,6) \in R_2 \Rightarrow (2,6) \in R_1 \circ R_2$.

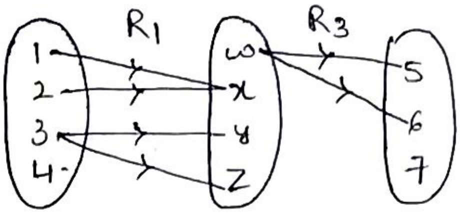
$\therefore \quad R_1 \circ R_2 = \{(1,6)\ (2,6)\}$. (see fig (i))



$R_1 \circ R_2$

There is no element $(a,b) \in R_1$ such that $(b,c) \in R_3$

$\therefore \quad R_1 \circ R_3 = \{\ \} = \phi$ (see fig (ii))



$R_1 \circ R_3$

ii) $M(R_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

$M(R_1 \circ R_2) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

$M(R_2) = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$

iii) consider $M(R_1) \cdot M(R_2) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

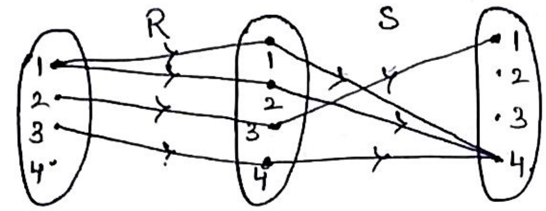
$= M(R_1 \circ R_2)$.

6) Let $A = \{1,2,3,4\}$ and $R = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\}$, $S = \{(3,1)(4,4)(2,4)(1,4)\}$ be relations on A. Determine the relations $R \circ S$, $S \circ R$, $R \circ (R \circ S)$, $R \circ (S \circ R)$, $S \circ (R \circ S)$, $S \circ (S \circ R)$, R^2 and S^2.

P.T.O.

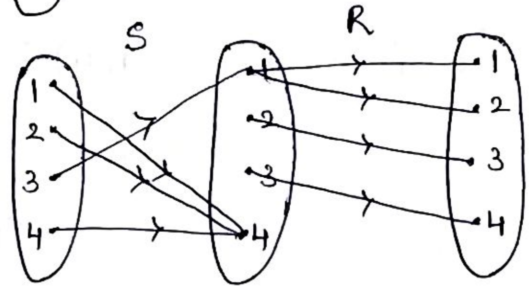
Soln :- $R \circ S = \{(1,4)(2,1)(3,4)\}$

$R \circ S$:



$S \circ R = \{(3,1)\ (3,2)\}$

$S \circ R$:



$R \circ (R \circ S) = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\} \circ \{(1,4)\ (2,1)\ (3,4)\}$

$= \{(1,4)\ (1,1)\ (2,4)\}$

$R \circ (S \circ R) = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\} \circ \{(3,1)\ (3,2)\}$

$= \{(2,1)\ (2,2)\}$

$S \circ (R \circ S) = \{(3,1)(4,4)(2,4)(1,4)\} \circ \{(1,4)\ (2,1)\ (3,4)\}$

$= \{(3,4)\}$

$S \circ (S \circ R) = \{(3,1)(4,4)\ (2,4)\ (1,4)\} \circ \{(3,1)\ (3,2)\}$

$= \phi . = \{\ \}$

$R^2 = R \circ R = \{(1,1)\ (1,2)\ (2,3)\ (3,4)\} \circ \{(1,1)\ (1,2)\ (2,3)\ (3,4)\}$

$= \{(1,1)\ (1,2)\ (1,3)\ (2,4)\}$

$S^2 = S \circ S = \{(3,1)\ (4,4)(2,4)\ (1,4)\} \circ \{(3,1)\ (4,4)\ (2,4)\ (1,4)\}$

$= \{(3,4)\ (4,4)\ (2,4)\ (1,4)\}$.

7) If $A = \{1,2,3,4\}$ and R is a relation on A defined by $R = \{(1,2)\ (1,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\}$. find R^2 and R^3. write down their digraphs. Find $M(R)$, $M(R^2)$, $M(R^3)$. verify that $M(R^2) = [M(R)]^2$ and $M(R^3) = [M(R)]^3$.

QP

Soln :- $R^2 = R \circ R = \{(1,2)\ (1,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\}$

$\circ$

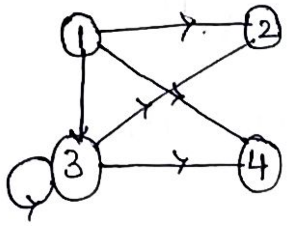
$\{(1,2)\ (1,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\}$

$\therefore \; R^2 = \{(1,4)\ (1,2)\ (1,3)\ (3,4)\ (3,2)\ (3,3)\}$.

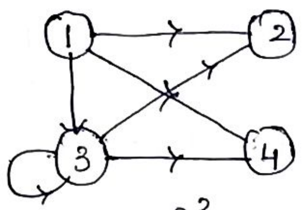
$$R^3 = R^2 \circ R = \{(1,4)(1,2)(1,3)(3,4)(3,2)(3,3)\}$$
$$\circ$$
$$\{(1,2)(1,3)(2,4)(3,2)(3,3)(3,4)\}$$

$$R^3 = \{(1,4)(1,2)(1,3),\ (3,4)(3,2)(3,3)\}.$$

The digraphs of R^2 and R^3 are as shown below:



R^2



R^3

$$M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(R^2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad M(R^3) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

by matrix multiplication with the stipulation that $1+1=1$, we find

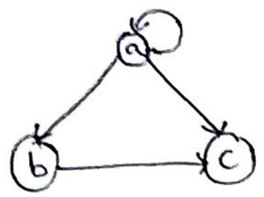
$$[M(R)]^2 = M(R) \cdot M(R) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

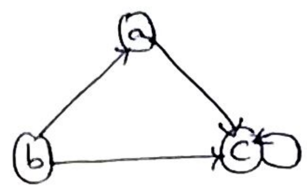
$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(R^2).$$

and $[M(R)]^3 = M(R) \cdot M(R^2) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = M(R^3).$$

8) The digraphs of two relations R and S on the set $A = \{a, b, c\}$ are given below. Draw the digraphs of $\overline{R}$, $R \cup S$, $R \cap S$ and R^c.

R: 

S: 

Soln :- From the digraphs,

$R = \{(a,a)\ (a,b)\ (a,c)\ (b,c)\}$ & $S = \{(a,c)\ (b,a)\ (b,c)\ (c,c)\}$.

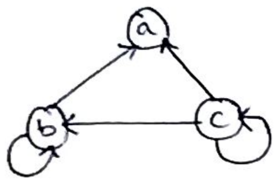
$\therefore \overline{R} = \{(b,a)\ (b,b)\ (c,a)\ (c,b)\ (c,c)\}$

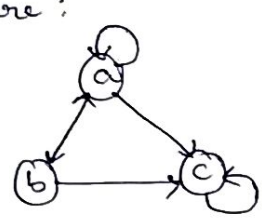
$R \cup S = \{(a,a)\ (a,b)\ (a,c)\ (b,a)\ (b,c)\ (c,c)\}$.

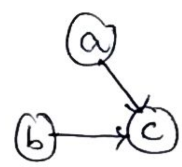
$R \cap S = \{(a,c)\ (b,c)\}$

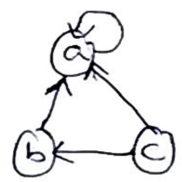
$R^c = \{(a,a)\ (b,a)\ (c,a)\ (c,b)\}$.

The digraphs of these are :

$\overline{R}$: 

$R \cup S$: 

$R \cap S$: 

R^c: 

9) For $A = \{1, 2, 3, 4\}$, let R and S be the relations on A defined by $R = \{(1,2)\ (1,3)\ (2,4)\ (4,4)\}$ and $S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (2,3)\ (2,4)\}$. find $R \circ S$, $S \circ R$, R^2, S^2 and write down their matrices.

QP

11) Let $R = \{(1,1)(1,2)(2,3)(3,3)(3,4)\}$ be a relation on $A = \{1,2,3,4\}$.

QP

i) Draw the digraph of R. ii) obtain R^2, R^3 and draw the digraph of R^2, R^3 iii) find $M(R^2)$, $M(R^3)$.

Properties of Relations :

1) Reflexive relation : A relation R on a set A is said to be reflexive if $(a,a) \in R$ $\forall a \in A$ (all a), i.e. aRa, $\forall a \in A$.

R is not reflexive (i.e non-reflexive) if $\exists$ some $a \in A$ such that $(a,a) \notin R$.

Ex :- 1) $\leq$, $=$, $\geq$ are reflexive relations on the set of all real no's, where as $<$, $>$ are not reflexive relations on the set of all real no's. ∵ $a = a$ for $\forall a \in R$

2) If $A = \{1,2,3,4\}$, then $R = \{(1,1)\ (2,2)\ (3,3)\}$ is not reflexive b'coz $4 \in A$ but $(4,4) \notin R$.

2) Irreflexive relation :- A relation R on a set A is said to be irreflexive if $(a,a) \notin R$, $\forall a \in A$ (all a). i.e there is no element of A related to itself.

Ex :- '$<$', '$>$' are irreflexive on the set of all real no's.

Note :- 1) A non reflexive relation need not be irreflexive.

2) A relation can be neither reflexive nor irreflexive.

3) Symmetric Relation :- A relation R on a set A is said to be symmetric whenever $(a,b) \in R$, $(b,a) \in R$ $\forall a,b \in A$.

4) Asymmetric relation :- A relation R on a set A is said to be asymmetric whenever $(a,b) \in R$, $(b,a) \notin R$ $\forall a,b \in A$.

5) Antisymmetric relation :- A relation R on a set A is said to be anti-symmetric if whenever $(a,b) \in R$ and $(b,a) \in R$ then $a = b$. i.e $(a,b) \in R$, and $a \neq b$ then $(b,a) \notin R$.

Ex :- '$\leq$' is antisymmetric on the set of all real numbers, b'coz if $a \leq b$ and $b \leq a$, then $a = b$.

Note : 1) Asymmetric and antisymmetric relations are not same.

2) A relⁿ can be both symmetric and antisymmetric. It can be neither symmetric nor antisymmetric.

6) <u>Transitive relation</u> :- A relation R on a set A is said to be transitive if whenever $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$ $\forall\ a,b,c \in A$.

Ex :- '$\leq$', '$\geq$' are transitive on the set of all real no's, b'coz if $a \leq b$ and $b \leq c$ then $a \leq c$ and if $a \geq b$ and $b \geq c$ then $a \geq c$. $\forall\ a,b,c \in \mathbb{R}$ (real no's)

<u>Note</u> :- R is not transitive if there exist $a,b,c \in A$ such that $(a,b) \in R$ and $(b,c) \in R$ but $(a,c) \notin R$.

— x —

<u>Equivalence relation</u> :- A relation R on a set A is said to be an equivalence relation on A if R is reflexive, symmetric and transitive on A.

<u>Ex</u> :- '$=$' is an equivalence relation on the set of all real no's.

<u>Equivalence classes</u> :- Let R be an equivalence relation on a set A and $a \in A$. Then the set of all those elements of A which are related to a by R is called the equivalence class of a with respect to R. This equivalence class is denoted by $R(a)$ or $[a]$ or $\bar{a}$. Thus $\bar{a} = [a] = R(a) = \{x \in A \mid (x,a) \in R\}$.

$\Downarrow$ xRa

Ex :- $R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,3)\}$ $[1] = \{1,2\}$ $[2] = \{1,2\}$ $[3] = \{3\}$

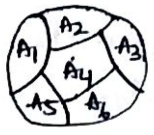
<u>Partition of a set</u> : Let A be a non empty set. Suppose there exist non empty subsets $A_1, A_2, A_3 \ldots A_K$ of A such that the following two conditions hold :

i) A is union of $A_1, A_2, \ldots A_K$ <u>ie</u> $A = A_1 \cup A_2 \cup \ldots \cup A_K$.

ii) Any 2 of the subsets $A_1, A_2 \ldots A_K$ are disjoint <u>ie</u> $A_i \cap A_j = \phi$ for $i \neq j$.

Then $P = \{A_1, A_2, \ldots, A_K\}$ is called a Partition of A. Also $A_1, A_2 \ldots A_K$ are called the blocks or cells of the Partition.

A Partition of a set with 6 blocks (cells) is as shown below:



Problems:

1) Let $A = \{1, 2, 3\}$. Determine the nature of the following relations on A.

i) $R_1 = \{(1,2)\ (2,1)\ (1,3)\ (3,1)\}$.
- Irreflexive
- symmetric
- Non-transitive.

ii) $R_2 = \{(1,1)\ (2,2)\ (3,3)\ (2,3)\}$.
- Reflexive
- Transitive
- Asymmetric (not symmetric)

iii) $R_3 = \{(1,1)\ (2,2)\ (3,3)\}$
- Reflexive
- symmetric.
- transitive

$\}$ Equivalence relation

iv) $R_4 = \{(1,1)\ (2,2)\ (3,3)\ (2,3)\ (3,2)\}$
- Reflexive
- symmetric
- Transitive

$\}$ Equivalence Relation

v) $R_5 = \{(1,1)\ (2,3)\ (3,3)\}$
- Non-reflexive or Irreflexive
- not symmetric i.e Asymmetric.
- Transitive

vi) $R_6 = \{(2,3)\ (3,4)\ (2,4)\}$.
- Transitive
- Irreflexive.
- Not symmetric

vii) $R_7 = \{(1,3)\ (3,2)\}$
- Irreflexive
- non-transitive
- not symmetric

P.T.O.

2) If $A = \{1, 2, 3, 4\}$ and R be a relation on A. Giver an example of a relation for each of the following :

i) reflexive and symmetric, but not transitive.

ii) reflexive and transitive, but not symmetric

iii) symmetric and transitive, but not reflexive.

Soln :- i) $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\ (2,1)\ (2,3)\ (3,2)\}$.

ii) $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\ (1,2)\}$

iii) $\{(1,1)\ (2,2)\ (1,2)\ (2,1)\}$.

3) Let $A = \{1, 2, 3, 4\}$ and $R = \{(1,1)(1,2)(2,1)(2,2)\ (3,4)(4,3)\ (3,3)\ (4,4)\}$ be a relation on A. Verify that R is an equivalence relation.

Soln :- To verify R is an equivalence relation, we have to show that R is reflexive, symmetric and transitive.

i) $(1,1)\ (2,2)\ (3,3)\ (4,4)$ belong to R. ie $(a,a) \in R \ \forall a \in A$.

$\therefore$ R is reflexive.

ii) $(1,2), (2,1) \in R$ and $(3,4)\ (4,3) \in R$

ie if $(a,b) \in R$, then $(b,a) \in R$ for $a, b \in A$.

$\therefore$ R is symmetric.

iii) $(1,2)\ (2,1)\ (1,1) \in R$, $(2,1)\ (1,2)\ (2,2) \in R$,

$(4,3)\ (3,4)\ (4,4) \in R$.

ie if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R \ \forall a, b, c \in A$

$\therefore$ R is transitive.

Thus R is an equivalence relation.

4) Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. On this set define the relation R by $(x,y) \in R$ iff $x-y$ is a multiple of 5.

QP

Verify that R is an equivalence relation.

Soln :- i) for any $x \in A$, we have $x - x = 0$ is a multiple of 5

($\because 0 = 5 \times 0$) ie $(x,x) \in R$.

$\therefore$ R is reflexive.

ii) for any $x, y \in A$,

if $(x,y) \in R$ then $x-y = 5k$ for some integer k.

$\Rightarrow$ $y - x = 5(-k)$. so that $(y,x) \in R$.

$\therefore$ R is symmetric.

iii) for any $x, y, z \in A$, if $(x,y) \in R$ and $(y,z) \in R$, then

$x-y = 5k_1$ and $y-z = 5k_2$ for some integers k_1 and k_2.

$$\therefore \quad x - z = (x-y) - (z-y)$$
$$= (x-y) + (y-z)$$
$$= 5k_1 + 5k_2 = 5(k_1+k_2)$$

$\therefore$ $(x,z) \in R$. $\Rightarrow$ R is Transitive.

Thus R is an equivalence relation.

5) If $A = A_1 \cup A_2 \cup A_3$, where $A_1 = \{1,2\}$, $A_2 = \{2,3,4\}$ and $A_3 = \{5\}$
Define the relation R on A by xRy iff x and y are in the same set A_i, $i = 1,2,3$. Is R an equivalence relation?.

Soln:- $A = A_1 \cup A_2 \cup A_3 = \{1,2,3,4,5\}$

$R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (2,3)\ (2,4)\ (3,2)\ (3,3)\ (3,4)\ (4,2)\ (4,3)$
$(4,4)\ (5,5)\}$

i) R is reflexive $\because$ $(a,a) \in R\ \forall\ a \in A$.

ii) R is symmetric $\because$ whenever $(a,b) \in R$, $(b,a) \in R$ $\forall\ a,b \in A$.

iii) R is not transitive $\because$ $(1,2) \in R$, $(2,3) \in R$ but $(1,3) \notin R$.

$\therefore$ R is not an equivalence relation.

6) for a fixed integer $n > 1$, Prove that the relation "Congruent modulo n" is an equivalence relation on the set of all integers $\mathbb{Z}$.

QP

Soln:- for any $a, b \in \mathbb{Z}$, "a is congruent to b modulo n" $[a \equiv b \pmod{n}]$
if $a-b$ is a multiple of n. ie $a - b = kn$ for some $k \in \mathbb{Z}$.

Let R be a relation on $\mathbb{Z}$ defined by aRb iff $a \equiv b \pmod{n}$

i) we have $a - a = 0 = 0 \times n$. $\Rightarrow$ $a \equiv a \pmod{n}$.

$\therefore$ aRa.

$\Rightarrow$ R is reflexive.

ii) since aRb, $a \equiv b \pmod n$

$\therefore\ a-b = kn.$

$\Rightarrow b-a = (-k)n.$

$\Rightarrow b \equiv a \pmod n \quad (\because k \in \mathbb{Z},\ -k \in \mathbb{Z}.)$

$\therefore\ bRa.$

Thus whenever aRb, bRa $\forall\, a,b \in \mathbb{Z}$.

$\therefore$ R is symmetric.

iii) Let aRb and bRc.

ie $a \equiv b \pmod n$ and $b \equiv c \pmod n$.

ie $a-b = k_1 n \rightarrow (1)$ and $b-c = k_2 n \rightarrow (2)$, $k_1, k_2 \in \mathbb{Z}$.

$(1)+(2) \Rightarrow a-c = k_1 n + k_2 n$

$a-c = (k_1+k_2)n.$

$\Rightarrow a \equiv c \pmod n \quad (\because k_1+k_2 \in \mathbb{Z})$

$\therefore\ aRc.$

Thus aRb and $bRc \Rightarrow aRc$, $\forall\, a,b,c \in \mathbb{Z}$.

$\therefore$ R is transitive.

Thus R is an equivalence relation.

7) For the equivalence relation, $R = \{(1,1)(1,2)(2,1)(2,2)(3,4)(4,3)(3,3)(4,4)\}$ defined on the set $A = \{1,2,3,4\}$. Determine the Partition induced.

Soln :- The equivalence classes of the elements of A w.r.t R are

$[1] = \{1,2\} \quad [2] = \{1,2\} \quad [3] = \{3,4\} \quad [4] = \{3,4\}.$

Of these, only $[1]$ and $[3]$ are distinct.

$\therefore$ Partition $P = \{[1],[3]\} = \{\{1,2\}\{3,4\}\}$ is the Partition of the given A induced by R.

[Observe that $A = [1] \cup [3] = \{1,2\} \cup \{3,4\} = \{1,2,3,4\}$]

P.T.O.

8) Find the Partition of A induced by R, given

$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$.

Sol^n:– $R = \{(x, y) \in R$ iff $x - y$ is a multiple of $5\}$.

i.e. $R = \{(1,1)\ (2,2) \dots (12,12)\ (6,1)\ (7,2)\ (8,3)\ (9,4)\ (10,5)\ (11,6)\ (12,7)\ (11,1)\ (12,2)\ (1,6)\ (2,7)\ (3,8)\ (4,9)\ (5,10)\ (6,11)\ (7,12)\ (1,11)\ (2,12)\}$. (or) [see ⊛].

∴ Equivalence classes are

$[1] = \{1, 6, 11\} = [6] = [11]$ $[2] = \{2, 7, 12\} = [7] = [12]$ $[3] = \{3, 8\} = [8]$

$[4] = \{4, 9\} = [9]$, $[5] = \{5, 10\} = [10]$.

All these classes are distinct.

∴ $P = \{[1], [2], [3], [4], [5]\}$ is the Partition of A induced by R, and $A = \{1, 6, 11\} \cup \{2, 7, 12\} \cup \{3, 8\} \cup \{4, 9\} \cup \{5, 10\}$.

9) Let $A = \{1, 2, 3, 4, 5, 6, 7\}$ and R be the equivalence relation on A that induces the partition $A = \{1, 2\} \cup \{3\} \cup \{4, 5, 7\} \cup \{6\}$. find R.

Sol^n:- Given Partition has 4 blocks : $\{1, 2\}$, $\{3\}$, $\{4, 5, 7\}$, $\{6\}$.

Let R be the equivalence relation inducing this Partition.

Since 1, 2 are in the same block, we have 1R1, 1R2, 2R1, 2R2

Since 3 belongs to block [3], 3R3.

Since 4, 5, 7 belongs to same block,

4R4, 4R5, 4R7, 5R4, 5R5, 5R7, 7R4, 7R5, 7R7.

Since 6 belongs to [6], 6R6.

∴ $R = \{(1,1)\ (1,2)\ (2,1)\ (2,2)\ (3,3)\ (4,4)\ (4,5)\ (4,7)\ (5,4)\ (5,5)\ (5,7)\ (7,4)\ (7,5)\ (7,7)\ (6,6)\}$.

⊛ Prob 8) by def^n $[a] = \{x \in A \mid x R a\}$.

∴ $[1] = \{x \in A \mid x R 1\} = \{x \in A \mid x - 1$ is a multiple of $5\}$

$= \{1, 6, 11\}$.

10) Let $A = \{1,2,3,4,5\}$. Define a relation R on $A \times A$ by $(x_1, y_1) R (x_2, y_2)$ if and only if $x_1 + y_1 = x_2 + y_2$.

Q8

i) verify that R is an equivalence relation on $A \times A$.

ii) Determine the equivalence classes $[(1,3)]$, $[(2,4)]$ and $[(1,1)]$.

iii) Determine the partition of $A \times A$ induced by R.

Soln:-

i) a) for any $(x,y) \in A \times A$, we have

$$x+y = x+y.$$

$\Rightarrow (x,y) R (x,y)$. $\Rightarrow$ R is reflexive.

b) for any $(x_1,y_1), (x_2,y_2) \in A \times A$,

Suppose $(x_1,y_1) R (x_2,y_2)$ then $x_1+y_1 = x_2+y_2$.

$\Rightarrow x_2+y_2 = x_1+y_1$.

$\Rightarrow (x_2,y_2) R (x_1,y_1)$

$\therefore$ R is symmetric.

c) for any $(x_1,y_1), (x_2,y_2), (x_3,y_3) \in A \times A$,

suppose $(x_1,y_1) R (x_2,y_2)$ and $(x_2,y_2) R (x_3,y_3)$

then $x_1+y_1 = x_2+y_2$ and $x_2+y_2 = x_3+y_3$

$\Rightarrow x_1+y_1 = x_3+y_3$.

$\Rightarrow (x_1,y_1) R (x_3,y_3)$.

$\therefore$ R is transitive.

Thus R is an equivalence relation.

ii) we have $[(1,3)] = \{(x,y) \in A \times A \mid (x,y) R (1,3)\}$

$= \{(x,y) \in A \times A \mid x+y = 1+3 = 4\}$

$= \{(1,3)\ (2,2)\ (3,1)\}$ $[\because A = \{1,2,3,4,5\}]$

III^{ly} $[(2,4)] = \{(1,5)\ (2,4)\ (3,3)\ (4,2)\ (5,1)\}$

$[(1,1)] = \{(1,1)\}$

iii) To determine the partition induced by R, we have to first find the equivalence classes of all elements (x,y) of $A \times A$ w.r.t R.

P.T.O.

we have $[(1,1)] = \{(1,1)\}$.

$[(1,2)] = \{(1,2)\ (2,1)\} = [(2,1)]$

$[(1,3)] = \{(1,3)\ (3,1), (2,2)\} = [(3,1)] = [(2,2)]$

$[(1,4)] = \{(1,4)\ (4,1)\ (2,3)\ (3,2)\} = [(4,1)] = [(2,3)] = [(3,2)]$

$\cancel{[(2,4)] = \{(2,4)\ (4,2)\ (3,3)}$

$[(1,5)] = \{(1,5)\ (5,1)\ (3,3)\ (2,4)\ (4,2)\} = [(5,1)] = [(3,3)] = [(2,4)] = [(4,2)]$

$[(2,5)] = \{(2,5)\ (5,2)\ (3,4)\ (4,3)\} = [(5,2)] = [(3,4)] = [(4,3)]$

$[(3,5)] = \{(3,5)\ (5,3)\ (4,4)\} = [(4,4)] = [(5,3)]$.

$[(4,5)] = \{(4,5)\ (5,4)\} = [(5,4)]$

$[(5,5)] = \{(5,5)\}$.

Thus $[(1,1)]$, $[(1,2)]$, $[(1,3)]$, $[(1,4)]$, $[(1,5)]$, $[(2,5)]$, $[(3,5)]$, $[(4,5)]$, $[(5,5)]$ are the only distinct equivalence classes of $A \times A$ on R.

$\therefore$ Partition of $A \times A$ induced by R is represented by

$P = \{[(1,1)], [(1,2)], [(1,3)], [(1,4)], [(1,5)], [(2,5)], [(3,5)], [(4,5)], [(5,5)]\}$.

$(A \times A = [(1,1)] \cup [(1,2)] \cup [(1,3)] \cup [(1,4)] \cup [(1,5)] \cup [(2,5)] \cup [(3,5)] \cup [(4,5)] \cup [(5,5)])$.

11) Find the number of equivalence relations that can be defined on a finite set A with $|A| = 6$.

QP

Soln:-

Note:- The no. of possible ways to assign 'm' distinct objects into 'n' identical places with empty places allowed is given by the formula $p(m) = \sum_{i=1}^{n} S(m,i)$ for $m \geq n$, where $S(m,n)$ is the Stirling number of 2nd kind given by

$$S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right) (n-k)^m \quad \text{for } m \geq n.$$

This no. represents the no. of ways of arranging 'm' objects into 'n' distinct containers with no container (Place) left empty.

Soln :- Since $|A| = 6$, the partition of A can have atmost 6 cells. Treating the elements of A as objects (ie $m = 6$) and cells as containers (ie $n = 6$), the no. of partitions having k cells is $S(6,k)$. Since k varies from 1 to 6, the total no. of different partitions of A is

$$p(6) = \sum_{i=1}^{6} S(6,i) = S(6,1) + S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6)$$

$$\therefore \quad S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^{n}C_{n-k}\right) (n-k)^m .$$

$$S(6,1) = \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \left({}^{1}C_{1-k}\right) (1-k)^6$$

$$= 1 . {}^{1}C_1 . 1^6 + (-1)\, {}^{1}C_0 \,(0)^6$$

$$= 1 + 0 = 1 .$$

$$S(6,2) = \frac{1}{2!} \sum_{k=0}^{2} (-1)^k \left({}^{2}C_{2-k}\right) (2-k)^6$$

$$= \frac{1}{2}\left[1 . 2^6 - {}^{2}C_1 . 1^6 + {}^{2}C_0 \,(0)^6\right] .$$

$$= \frac{1}{2}\left[2^6 - 2\right] = 31 .$$

$$S(6,3) = \frac{1}{3!} \sum_{k=0}^{3} (-1)^k \left({}^{3}C_{3-k}\right) (3-k)^6$$

$$= \frac{1}{6}\left[3^6 - 3 \times 2^6 + 3 \times 1^6\right] = 90 .$$

$$S(6,4) = \frac{1}{4!} \sum_{k=0}^{4} (-1)^k \left({}^{4}C_{4-k}\right) (4-k)^6$$

$$= \frac{1}{24}\left[4^6 - 4 \times 3^6 + 6 \times 2^6 - 4 \times 1^6\right] = 65$$

$$S(6,5) = \frac{1}{5!} \sum_{k=0}^{5} (-1)^k \left({}^{5}C_{5-k}\right) (5-k)^6$$

$$= \frac{1}{120}\left[5^6 - 5 \times 4^6 + 10 \times 3^6 - 10 \times 2^6 + 5 \times 1^6\right] = 15$$

$$S(6,6) = \frac{1}{6!} \sum_{k=0}^{6} (-1)^k \left({}^{6}C_{6-k}\right) (6-k)^6$$

$$= \frac{1}{720}\left[6^6 - 6 \times 5^6 + 15 \times 4^6 - 20 \times 3^6 + 15 \times 2^6 - 6 \times 1^6 + 0\right] = 1$$

$\therefore$ No. of Partitions of A is

$P(6) = 1 + 31 + 90 + 65 + 15 + 1 = 203$ ie 203 equivalence relations can be defined on A.

12) Let $A = \{1,2,3,4\}$, $R = \{(1,3)\ (1,1)\ (3,1)\ (1,2)\ (3,3)\ (4,4)\}$

QP be the relation on A. Determine whether the relation R is reflexive, irreflexive, symmetric, antisymmetric or transitive.

Soln:- i) R is not reflexive $\because (2,2) \notin R$.

ii) R is not irreflexive since $(1,1), (3,3)\ (4,4) \in R$.

iii) R is not symmetric (or R is asymmetric) $\because (1,2) \in R$, but $(2,1) \notin R$.

iv) R is not antisymmetric $\because (1,3) \in R$ and $1 \neq 3$, but $(3,1) \in R$.

v) R is not transitive $\because (3,1) \in R$, $(1,2) \in R$ but $(3,2) \notin R$.

13) Let S be the set of all non-zero integers and $A = S \times S$

QP on A. Define the relation R by $(a,b)\, R\, (c,d)$ iff $ad = bc$.

Show that R is an equivalence relation.

Soln:- i) for any $a \in S$, $(a,a)\, R\, (a,a)$ b'coz $aa = aa$.

$\therefore$ R is reflexive on A.

ii) suppose $(a,b)\, R\, (c,d)$, then $ad = bc$

$\Rightarrow da = cb$

$\Rightarrow cb = da$

$\Rightarrow (c,d)\, R\, (a,b)$

$\therefore$ R is symmetric on A.

iii) Suppose that $(a,b)\, R\, (c,d)$ and $(c,d)\, R\, (e,f)$, then

$ad = bc$ and $cf = de$.

$\Rightarrow c = \frac{de}{f}$

$\therefore a\cancel{d} = b \times \frac{\cancel{d}e}{f} \Rightarrow af = be$

$\therefore (a,b)\, R\, (e,f) \Rightarrow$ R is transitive on A.

$\therefore$ R is an equivalence relation.

14) Let N be the set of all natural numbers. On $N \times N$,
QS the relation R is defined by $(a, b) R (c, d)$ iff

$a+d = b+c$. Show that R is an equivalence relation.

Find the equivalence class of the element $(2,5) \in N \times N$.

Soln:- i) For any $a \in N$, $(a,a) R (a,a)$ $\because$ $a+a = a+a$.

$\therefore$ R is reflexive.

ii) Suppose $(a,b) R (c,d)$ $\therefore$ $a+d = b+c$

$\Rightarrow$ $b+c = a+d$

$\Rightarrow$ $c+b = a+d$.

$\Rightarrow$ $(c, d) R (a,b)$.

$\therefore$ R is symmetric.

iii) Suppose $(a,b) R (c,d)$ and $(c,d) R (e,f)$, then

$a+d = b+c$ and $c+f = d+e$.

$\Rightarrow$ $c = d+e-f$.

$\therefore$ $a+\cancel{d} = b+\cancel{d}+e-f$

$\Rightarrow$ $a+f = b+e$

$\Rightarrow$ $a+f = e+b$

$\Rightarrow$ $(a,b) R (e,f)$. $\therefore$ R is transitive.

Thus R is an equivalence relation.

To find equivalence class of $(2,5)$:

$[(2,5)] = \{(x,y) \mid (x,y) R (2,5)\}$ $(\because a+d = b+c)$

$= \{(x,y) \mid x+5 = y+2\}$

$= \{(x,y) \mid x-y = 2-5\} = \{(x,y) \mid x-y = -3\}$

$= \{(x,y) \mid y-x = +3\}$

$= \{(1,4) (2,5) (3,6) (4,7) \ldots\}$.

P.T.O.

15) Let R be an equivalence relation on set A and
QP $a, b \in A$. Then prove the following are equivalent :

i) $a \in [a]$ ii) aRb iff $[a] = [b]$

iii) if $[a] \cap [b] \neq \phi$, then $[a] = [b]$.

Soln :- i) Since R is reflexive, we have aRa.

$\therefore$ $a \in [a]$.

ii) Suppose aRb. Take any $x \in [a]$, then xRa.

$\therefore$ we have xRa and aRb.

$\Rightarrow$ xRb (by transitive property)

$\Rightarrow$ $x \in [b]$.

Thus $[a] \subseteq [b]$.

IIIly we find that $[b] \subseteq [a]$.

$\therefore$ $[a] = [b]$.

Conversely, suppose $[a] = [b]$

Since $a \in [a]$ (from (i))

$\Rightarrow$ $a \in [b]$

Thus aRb.

iii) Suppose $[a] \cap [b] \neq \phi$.

$\Rightarrow$ $\exists$ $x \in A$ such that $x \in [a]$ and $x \in [b]$.

$\Rightarrow$ xRa and xRb.

$\Rightarrow$ aRx and xRb (by symmetry)

$\Rightarrow$ aRb (by transitivity)

$\Rightarrow$ $[a] = [b]$ (by (ii))

Partial Orders :- A relation R on a set A is said to be a partial order on A, if R is reflexive, antisymmetric and transitive on A. A set A with a partial order defined on it is called a Partially order set (or) a <u>Poset</u> and is denoted by the Pair (A, R).

Ex :- The relation '$\leq$' on the set of all integers is a Partial order. $\therefore (Z, \leq)$ is a Poset. The relation '$\geq$' on the set of all integers is a partial order. $\therefore (Z, \geq)$ is a Poset.

Total order : Let R be a partial order on a set A. Then R is called a total order on A, if for all $x, y \in A$, either xRy or yRx. In this case, the Poset (A, R) is called a totally ordered set.

Ex :- The Partial order relation '$\leq$' is a total order on the set of all real numbers $\mathbb{R}$ because for any $x, y \in \mathbb{R}$, we have $x \leq y$ or $y \leq x$. $\therefore (\mathbb{R}, \leq)$ is a totally ordered set.

Note :- Every total order is a partial order but every partial order need not be a total order.

Hasse Diagrams :- For a Partial order relation on a finite set, we can draw digraph of a Partial order.

1. Since a Partial order is reflexive, at every vertex of digraph of Partial order, there will be a loop. while drawing the digraph of partial order, we do not show the loops explicitly. They will be automatically understood by convention.
2. In the digraph of Partial order, if there is an edge from vertex a to b and an edge from vertex b to c, then there will be an edge from a to c (b'coz of transitivity). But we do not exhibit an edge from a to c explicitly. It will be automatically understood by convention.

3. To simplify the format of the digraph of a partial order, we represent the vertices by dots and draw the digraph in such a way that all edges point upward. With this convention we need not put arrows in the edges.

The digraph of a partial order drawn by adopting the conventions indicated above is called a Poset diagram or the Hasse digram for the Partial order.

Problems:-

1) Let $A = \{1,2,3,4\}$ and $R = \{(1,1)(1,2)(2,2)(2,4)(1,3)(3,3)(3,4),$ $(1,4)(4,4)\}$. Verify that R is a Partial order on A. Also write down the Hasse diagram for R.

Soln:- i) we see that $(a,a) \in R$ $\forall$ $a \in A$. Hence R is reflexive on A.

ii) if $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$, $\forall$ $a,b \in A$.

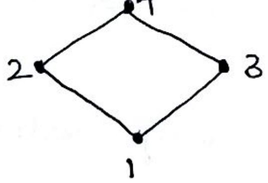
$\therefore$ R is antisymmetric.

iii) if $(a,b) \in R$ and $(b,c) \in R$ then we see that $(a,c) \in R$.

$\therefore$ R is transitive.

Thus R is a partial order on A. i.e (A,R) is a Poset.

The Hasse Diagram for R is as shown below:



→ No Δ^{le} shud come in Hasse diagram.

→ 2 no's which are related to eachother shud not be in same lin

2) Let R be a relation on the set $A = \{1,2,3,4\}$ defined by xRy iff x divides y. Prove that (A,R) is a Poset.

Q) Draw its Hasse diagram.

Soln:. $R = \{(x,y) \mid x,y \in A$ and x divides $y\}$

i.e $R = \{(1,1)(1,2)(1,3)(1,4)(2,2)(2,4)(3,3)(4,4)\}$.

i) we see that $(a,a) \in R$ $\forall$ $a \in A$. $\Rightarrow$ R is reflexive on A.

ii) If $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$.

$\therefore$ R is antisymmetric on A.

iii) If $(a,b) \in R$ and $(b,c) \in R$, then we see that $(a,c) \in R$.

$\therefore$ R is transitive.

Thus R is a partial order on A. i.e. (A, R) is a Poset.

The Hasse diagram for R is as shown below:

3) Let $A = \{1, 2, 3, 4, 6, 8, 12\}$. On A, define the partial ordering relation R by xRy iff $x \mid y$. Prove that R is a partial order on A. Draw the Hasse diagram for R.

Soln:. $R = \{(1,1)(1,2)(1,3)(1,4)(1,6)(1,8)(1,12)(2,2)(2,4)(2,6)(2,8)(2,12),$
$(3,3)(3,6)(3,12)(4,4)(4,8)(4,12)(6,6)(6,12)(8,8)(12,12)\}$.

i) we see that $(a,a) \in R \ \forall a \in A$. $\Rightarrow$ R is reflexive.

ii) if $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$.

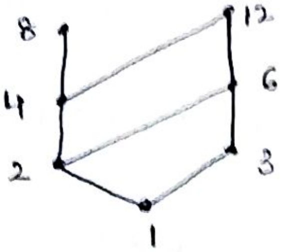
$\therefore$ R is antisymmetric.

iii) if $(a,b) \in R$ and $(b,c) \in R$, then we see that $(a,c) \in R$.

$\therefore$ R is transitive.

Thus R is a partial order on A i.e. (A, R) is a Poset.

The Hasse diagram is as below:



4) Draw the Hasse diagram representing the positive divisors of 36.

Soln:- The set of all positive divisors of 36 are:

$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$.

The relation R of divisibility i.e. aRb iff a divides b. is a Partial order on this set.

we note that, under R,

1 is related to all elements of D_{36}.

2 is " ——— 2, 4, 6, 12, 18, 36

3 " ——— 3, 6, 9, 12, 18, 36.

4 " ——— 4, 12, 36.

6 " ——— 6, 12, 18, 36.

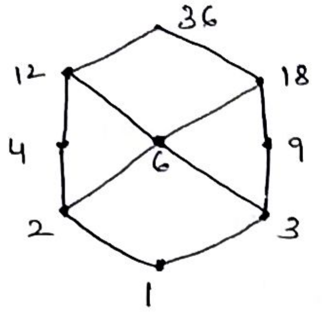
9 " ——— 9, 18, 36

12 " ——— " 12, 36

18 " ——— " 18, 36

36 " ——— 36.

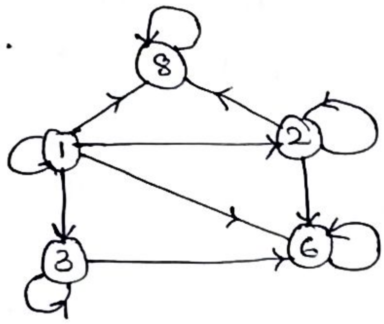
Hasse diagram :-



1 – least element.

36 – greatest element.

5) The digraph for a relation on the set $A = \{1, 2, 3, 6, 8\}$ is as shown below: Verify that (A, R) is a Poset and write down its Hasse diagram.



Soln:- $R = \{(1,1)(1,2)(1,3)(1,6)(1,8)(2,2)(2,6)(2,8)(3,3)(3,6)(6,6)(8,8)\}$.

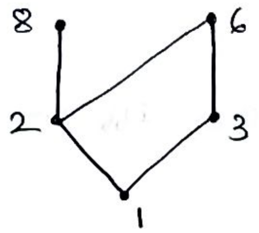
i) R is reflexive b'coz $(a,a) \in R \ \forall a \in A$.

ii) R is antisymmetric b'coz if $(a,b) \in R$ and $a \neq b$, then we see that $(b,a) \notin R$.

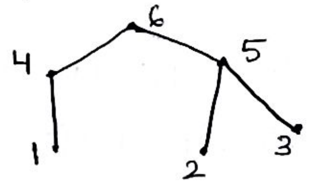
iii) R is transitive b'coz if $(a,b) \in R$ and $(b,c) \in R$, we see that $(a,c) \in R$.

Thus R is a partial order on A. i.e. (A,R) is a Poset.

Hasse diagram :



6) The Hasse diagram of a partial order R on the set $A = \{1,2,3,4,5,6\}$ is as given below. Write down R as a subset of $A \times A$. construct its digraph. Also determine the matrix of the Partial order.

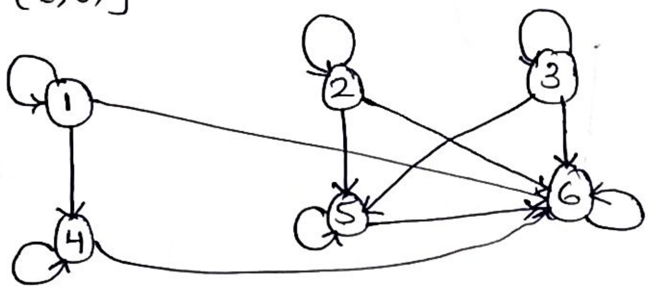


Soln :- from the diagram,

$1R1, 1R4, 1R6, 2R2, 2R5, 2R6,$
$3R3, 3R5, 3R6, 4R4, 4R6, 5R5, 5R6, 6R6.$

$\therefore R = \{(1,1)(1,4)(1,6)(2,2)(2,5)(2,6)(3,3)(3,5)(3,6)(4,4)(4,6)(5,5)(5,6)(6,6)\}$

Digraph :-

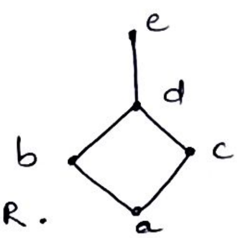


$$M(R) = M_R = \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

7) For $A = \{a,b,c,d,e\}$, the Hasse diagram for the Poset - (A, R) is as shown below:

QP

i) Determine the relation matrix for R.

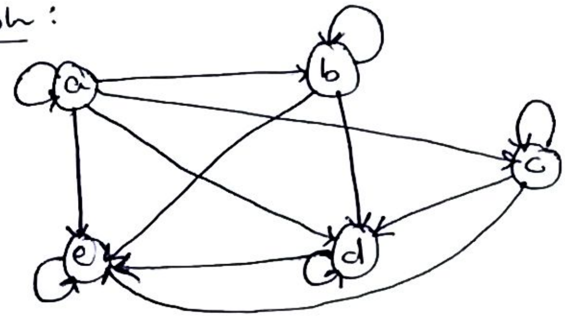


ii) Construct the digraph for R.

Soln :- $R = \{(a,a)(a,b)(a,c)(a,d)(a,e)(b,b)(b,d)(b,e)(c,c)(c,d)(c,e)(d,d)(d,e)(e,e)\}$.

i) $$M(R) = \begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

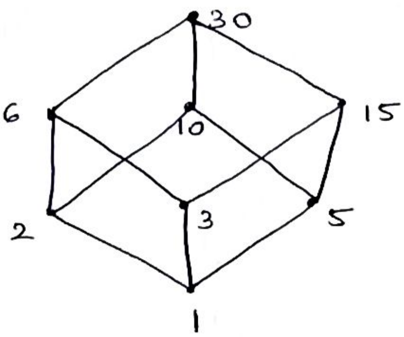
ii) Digraph :



8) In the following cases, consider the partial order of divisibility on the set A. Draw the Hasse diagram for the poset and determine whether the poset is totally ordered or not.

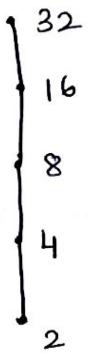
(i) $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ ii) $A = \{2, 4, 8, 16, 32\}$.

Soln :- i) $R = \{(1,1)(1,2)(1,3)(1,5)(1,6)(1,10)(1,15)(1,30)(2,2)(2,6)(2,10)(2,30)$
$(3,3)(3,6)(3,15)(3,30)(5,5)(5,10)(5,15)(5,30)(6,6)(6,30)$
$(10,10)(10,30)(15,15)(15,30)(30,30)\}$



(A, R) is not totally ordered set because if we consider 2, 3 in A neither 2 divides 3 nor 3 divides 2.

ii) $R = \{(2,2)(2,4)(2,8)(2,16)(2,32)(4,4)(4,8)(4,16)(4,32)(8,8)$
$(8,16)(8,32)(16,16)(16,32)(32,32)\}$.



(A, R) is totally ordered set.

<u>Extremal elements in Poset</u> : Consider a Poset (A, R), we define some special elements called extremal elements that may exist in A.

1) An element $a \in A$ is called a <u>maximal element</u> if there exists <u>no</u> x in A other than 'a' such that aRx ie 'a' is maximal element if and only if in Hasse diagram of R, no edge starts at 'a'. (bottom → top)

2) An element $a \in A$ is called a <u>minimal element</u> if there exists <u>no</u> x in A other than 'a' such that xRa ie 'a' is minimal element if and only if in Hasse diagram of R, no edge terminates at 'a'.

3) An element $a \in A$ is called a <u>greatest element</u> of A if xRa $\forall x \in A$. ie all elements of A shud be related to 'a'.

4) An element $a \in A$ is called a <u>least element</u> of A if aRx $\forall x \in A$ ie 'a' shud be related to all elements of A.

5) Let $B \subseteq A$. Then an element $a \in A$ is called an <u>upper bound</u> of B if xRa, $\forall x \in B$. ie all elements of subset B shud be related to $'a' \in A$

6) Let $B \subseteq A$. Then an element $a \in A$ is called a <u>lower bound</u> of B if aRx, $\forall x \in B$. ie $a \in A$ shud be related to all elements of subset B.

7) Let $B \subseteq A$. Then an element $a \in A$ is called the <u>Least Upper bound</u> (LUB) of B if

i) a is an upper bound of B and

ii) If a' is also an upper bound of B, then aRa'

8) Let $B \subseteq A$. Then an element $a \in A$ is called the <u>greatest lower bound</u> (GLB) of B if

i) a is a lower bound of B and

ii) If a' is also a lower bound of B, then $a'Ra$.

<u>Note</u>:- LUB is also called Supremum and
GLB is " ———— Infimum.

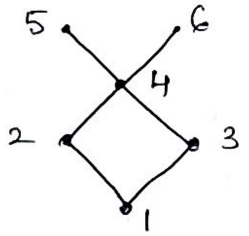
Lattices :- Let (A,R) be a Poset. This is called a Lattice if $\forall\ x, y \in A$, the elements LUB $\{x,y\}$ and GLB $\{x,y\}$ exist in A.

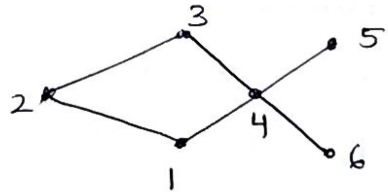
Ex :- 1) $(N, \leq)$ is a lattice. for all x,y in N,
GLB$\{x,y\}$ = min $\{x,y\}$ and LUB $\{x,y\}$ = max $\{x,y\}$.
Both of these belong to N.

2) $(Z^+, |)$ is a lattice where $|$ is the divisibility relation.
For all x,y in Z^+, GLB$\{x,y\}$ = gcd(x,y) and
LUB$\{x,y\}$ = lcm(x,y). Both of these belong to Z^+.

Problems :-

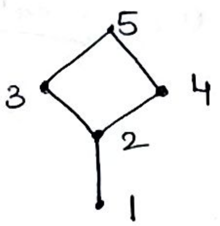
1) For the Poset (A, R) represented by the Hasse diagram, find
i) maximal ii) minimal iii) greatest and iv) least element(s)

a)


b)


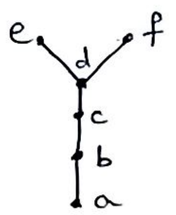
Soln :- a) 5 and 6 are maximal elements (∵ no edge starts at 5, 6)
1 is a minimal element (∵ no edge terminates at 1)
1 is the least element (∵ 1 is related to all elements of A)
No greatest element (∵ not 5 ∵ 6 is not related to 5
not 6 ∵ 5 '' —— 6

b) Maximal : 3, 5 Greatest : No greatest (∵ $5 \not R 3$ and $3 \not R 5$)
Minimal : 1, 6 Least : No least (∵ $1 \not R 6$ & $6 \not R 1$)

c)


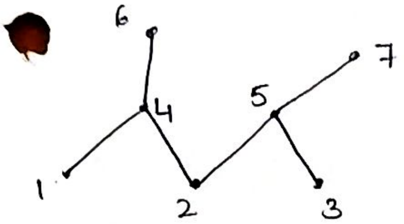
Maximal : 5
Minimal : 1

Greatest : 5 (∵ 1R5, 2R5, 3R5, 4R5, 5R5)
Least : 1 (∵ 1R1, 1R2, 1R3, 1R4, 1R5)

d)


Maximal : e, f
Minimal : a

greatest : No
least : a

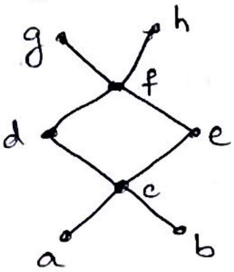
e) 

Maximal : 6, 7 greatest : No

Minimal : 1, 2, 3 least : No.

2) Consider the Hasse diagram of a Poset (A, R) given below:

SP If B = {c, d, e}, find (if they exist)



i) all upper bounds of B.

ii) all lower bounds of B

iii) the least upper bound of B

iv) the greatest lower bound of B.

soln :- i) All of c, d, e which are in B are related to f, g, h.

∴ f, g, h are upper bounds of B.

ii) The elements a, b, c are related to all of c, d, e which are in B. ∴ a, b, c are lower bounds of B.

iii) The upper bound f of B is related to the other upper bounds g and h of B. ∴ f is the LUB of B. i.e LUB(B) = f

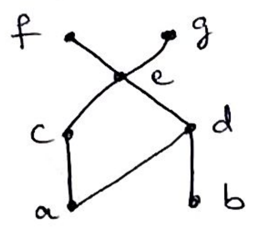
iv) The lower bounds a and b of B are related to the lower bound c of B. ∴ c is the GLB of B. i.e GLB(B) = c

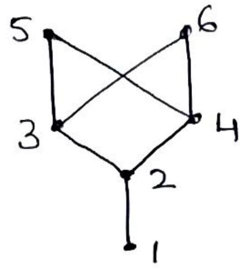
3) For the Posets shown in the following Hasse diagrams, find

i) all upper bounds ii) all lower bounds

iii) LUB and iv) GLB of the set B, where

B = {c, d, e} in case (a) and B = {3, 4, 5} in case (b).

(a) 

(b) 

soln:- (a) i) All of c, d, e which are in B are related to e, f, g.

∴ e, f, g are upper bounds of B.

ii) The element a is related to all of c, d, e which are in B. ∴ a is a lower bound of B.

iii) The upper bound e of B is related to the other upper bounds f and g of B.

$\therefore$ e is the LUB of B. i.e. LUB(B) = e.

iv) NO GLB(B) ~~exist~~.

b) i) 5 is ~~are~~ the upper bounds of B. (not 6 $\because$ 5 $\not R$ 6)

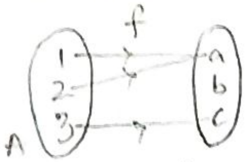
ii) 1, 2 are the lower bounds of B.

iii) There is no LUB of B.

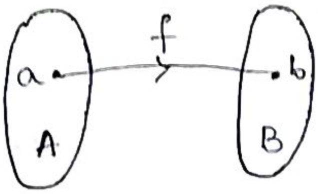
iv) GLB(B) = 2.

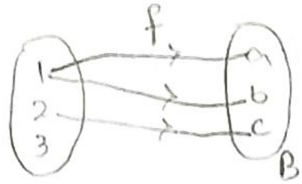
Functions

Def^n :- Let A and B be non empty sets. Then a function (or a mapping) from A to B is a relation from A to B such that for each a in A, there exists a unique b in B such that $(a,b) \in f$. Then we write $b = f(a)$. Here b is called the image of a and a is called the preimage of b, under f. The element a is also called an argument of the function f and $b = f(a)$ is called the value of the function f for the argument a. A function f from A to B is denoted by $f : A \rightarrow B$. The Pictorial representation of f is as below :



a fn.





is a rel^n not a fn.

"Every function is a relation but every relation is not a fn".

B'coz, if R is a relation from A to B then an element of A can be related to 2 elements of B under R. But under a function, an element of A can be related to only one element of B.

For the function $f : A \rightarrow B$, A is called the domain of f and B is called the co-domain of f. The subset of B consisting of the images of all elements of A under f is called the range of f, denoted by $f(A)$.

Note :-

1) Every a in A belong to some pair $(a,b) \in f$ and if $(a,b_1) \in f$ and $(a,b_2) \in f$ then $b_1 = b_2$.
2) An element $b \in B$ need not have a preimage in A, under f
3) Two different elements of A can have the same image in B, under f.

4) The statements $(a,b) \in f$, afb and $b = f(a)$ are equivalent.

5) If g is a function from A to B, then $f = g$ iff $f(a) = g(a)$, $\forall\, a \in A$.

6) The range of $f: A \to B$ is given by $f(A) = \{f(x) / x \in A\}$ and $f(A)$ is a subset of B.

7) For $f: A \to B$, if $A_1 \subseteq A$ and $f(A_1)$ is defined by $f(A_1) = \{f(x) / x \in A_1\}$, then $f(A_1) \subseteq f(A)$. Here $f(A_1)$ is called the image of A_1, under f.

8) for $f: A \to B$, if $b \in B$ and $f^{-1}(b)$ is defined by $f^{-1}(b) = \{x \in A \mid f(x) = b\}$, then $f^{-1}(b) \subseteq A$. Here $f^{-1}(b)$ is called the preimage of b, under f.

9) For $f: A \to B$, if $B_1 \subseteq B$ and $f^{-1}(B_1)$ is defined by $f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$, then $f^{-1}(B_1) \subseteq A$. Here $f^{-1}(B_1)$ is called the Preimage of B_1, under f.

Types of functions :-

1) Identity function :- A f^n f on a set A is an identity function if the image of every element of A (under f) is itself and is denoted by I_A.

only, not $A \to B$

i.e $f: A \to A$ is such that $f(a) = a$ $\forall\, a \in A$.

In case of identity function, $f(A) = A$.

2) Constant function :- A f^n $f: A \to B$ is called a constant f^n if $f(a) = c$ $\forall\, a \in A$. i.e f is a constant f^n if image of every element of A is same in B, and in this case $f(A) = \{c\}$.

3) onto function :- (surjective fn) A fn $f : A \to B$ is said to be an onto fn from A to B if for every element b of B, there exists an element a of A such that $f(a) = b$.

ié, f is an onto fn from A to B if every element b of B has a preimage in A. ié range of $f = B$.

No element in B shud be free.

4) One-to-one function :- (Injective fn) :- A fn $f : A \to B$ is said to be an one-to-one fn from A to B if different elements of A have different images in B under f.

ié if $f(a_1) = f(a_2)$ then $a_1 = a_2$ where $a_1, a_2 \in A$.

(or) if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$ (by contrapositive).

5) One-to-one correspondence (Bijective fn or Bijection) :- A fn $f : A \to B$ is said to be an one-to-one correspondence if f is both one-to-one and onto. If $f : A \to B$ is a bijective fn, then every element of A has a unique image in B and every element of B has a unique preimage in A.

Properties of functions :-

1) Let $f : X \to Y$ be a fn and A and B be arbitrary non-empty sub sets of X, then

i) If $A \subseteq B$, then $f(A) \subseteq f(B)$

ii) $f(A \cup B) = f(A) \cup f(B)$.

iii) $f(A \cap B) \subseteq f(A) \cap f(B)$ and the equality holds if f is 1-1.

2) Let A and B be finite sets and f be a fn from A to B, then the following are true :

i) If f is one-to-one then $|A| \leq |B|$.

ii) If f is onto then $|B| \leq |A|$.

iii) If f is bijective then $|A| = |B|$.

3) Suppose $|A| = |B|$ and $f : A \to B$, then f is one-to-one iff f is onto.

4) If $f : A \to B$ and $|A| = |B|$, then f is bijective iff f is one-to-one or onto.

5) If $f : A \to B$, $|A| = m$ and $|B| = n$ then there are n^m functions from A to B and if $m \le n$ then there are $\frac{n!}{(n-m)!}$ one-to-one functions from A to B.

Stirling number of second kind :-

Let A and B be finite sets with $|A| = m$ and $|B| = n$, where $m \geq n$. Then the number of onto functions from A to B is given by the formula :

$$p(m,n) = \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right) (n-k)^m$$

(If $m < n$, then there are no onto fn from A to B)

with $p(m,n)$ given by the above formula, the no. $\left[\frac{p(m,n)}{n!}\right]$ is called the Stirling number of the second kind and is denoted by $S(m,n)$.

i) The no. of possible ways to assign 'm' distinct objects to 'n' identical places (containers) with no place (container) left empty is given by

$$S(m,n) = \frac{p(m,n)}{n!} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \left({}^nC_{n-k}\right) (n-k)^m, \text{ for } m \geq n.$$

ii) The no. of possible ways to assign 'm' distinct objects to 'n' identical places with empty places allowed is given by

$$p(m) = \sum_{i=1}^{n} S(m,i), \quad \text{for } m \geq n.$$

Problems :

1) Let $A = \{1,2,3,4\}$. Determine whether or not the following relations on A are fns :

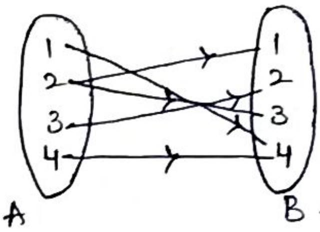
i) $f = \{(2,3)\ (1,4)\ (2,1)\ (3,2)\ (4,4)\}$

ii) $g = \{(2,1)\ (3,4)\ (1,4)\ (2,1)\ (4,4)\}$. If it is a fn, find its range

soln :-

i) we see that $(2,3) \in f$ and $(2,1) \in f$ i.e. the element 2 is related to two different elements 3 and 1, under f.

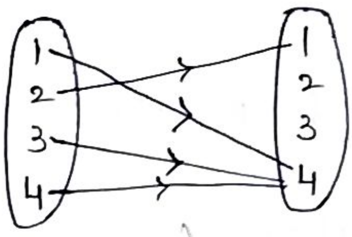
$\therefore$ f is not a fn.



ii) we see that, under g, every element of A is related to a unique element of A.

$\therefore$ g is a fn from A to A.

Range of $g = g(A) = \{1,4\}$



((2,1) appears twice. This has no special significance)

2) Let $A = \{0, \pm 1, \pm 2, \pm 3\}$. consider $f : A \to \mathbb{R}$ (where $\mathbb{R}$ is the set of all real numbers) defined by $f(x) = x^3 - 2x^2 + 3x + 1$ for $x \in A$. Find the range of f.

soln :-

$f(0) = 1.$

$f(1) = 1^3 - 2(1)^2 + 3(1) + 1 = 3$

|||ly $f(2) = 7$, $f(3) = 19$, $f(-1) = -5$, $f(-2) = -21$, $f(-3) = -53$.

$\therefore$ Range of $f = \{-53, -21, -5, 1, 3, 7, 19\}$.

3) Let $A = \{1,2,3,4,5,6\}$ and $B = \{6,7,8,9,10\}$. If a fn $f : A \to B$ is defined by $f = \{(1,7)\ (2,7)\ (3,8)\ (4,6)\ (5,9)\ (6,9)\}$.

Determine $f^{-1}(6)$ and $f^{-1}(9)$. If $B_1 = \{7,8\}$ and $B_2 = \{8,9,10\}$ find $f^{-1}(B_1)$ and $f^{-1}(B_2)$.

Soln :- By defn,

$f^{-1}(b) = \{x \in A \mid f(x) = b\}$

$\therefore\ f^{-1}(6) = \{x \in A \mid f(x) = 6\} = \{4\}$.

$f^{-1}(9) = \{x \in A \mid f(x) = 9\} = \{5, 6\}$.

For any $B_1 \subseteq B$, by defn

$f^{-1}(B_1) = \{x \in A \mid f(x) \in B_1\}$.

for $B_1 = \{7, 8\}$, $f(x) \in B_1$ when $f(x) = 7$ and $f(x) = 8$.

from the defn of f, we see that $f(x) = 7$ when $x = 1$ & $x = 2$,

and $f(x) = 8$ when $x = 3$.

$\therefore\ f^{-1}(B_1) = \{1, 2, 3\}$.

IIIly $f^{-1}(B_2) = \{x \in A \mid f(x) \in B_2\} = \{3, 5, 6\}$.

4) Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} 3x - 5 & \text{for } x > 0 \\ -3x + 1 & \text{for } x \leq 0. \end{cases}$

Q:

i) Determine $f(0)$, $f(-1)$, $f\left(\frac{5}{3}\right)$, $f(-5/3)$

ii) find $f^{-1}(0)$, $f^{-1}(1)$, $f^{-1}(-1)$, $f^{-1}(3)$, $f^{-1}(-3)$, $f^{-1}(-6)$.

iii) what are $f^{-1}([-5, 5])$ and $f^{-1}([-6, 5])$?.

soln :- i) $f(0) = -3(0) + 1 = 1$.

$f(-1) = -3(-1) + 1 = 4$.

$f(5/3) = 3\left(\frac{5}{3}\right) - 5 = 0$.

$f(-5/3) = -3\left(\frac{-5}{3}\right) + 1 = 6$.

ii) By defn $f^{-1}(b) = \{x \in \mathbb{R} \mid f(x) = b\}$.

a) $f^{-1}(0) = \{x \in \mathbb{R} \mid f(x) = 0\}$.

Consider $f(x) = 0$.

$3x - 5 = 0 \quad (x > 0)$ and $-3x + 1 = 0 \quad (x \leq 0)$

$3x = 5$; $3x = 1$

$x = 5/3 > 0$ possible. ; $x = 1/3 > 0$ # not possible.

Thus $f^{-1}(0) = \{5/3\}$.

b) $f^{-1}(1) = \{x \in \mathbb{R} \mid f(x) = 1\}$

$f(x) = 1 \Rightarrow 3x - 5 = 1 \quad (x > 0)$ and $-3x + 1 = 1 \quad (x \leq 0)$

$\Rightarrow 3x = 6$; $3x = 0$

$x = 2 > 0$ ✓ ; $x = 0$ ✓

$\therefore f^{-1}(1) = \{0, 2\}$.

c) $f^{-1}(-1) = \{x \in \mathbb{R} \mid f(x) = -1\}$

$f(x) = -1 \Rightarrow 3x - 5 = -1 \quad (x > 0)$ and $-3x + 1 = -1 \quad (x \leq 0)$

$\Rightarrow 3x = 4$; $-3x = -2$

$x = 4/3$ ✓ ; $x = 2/3$ ✗.

$\therefore f^{-1}(-1) = \{4/3\}$.

d) $f^{-1}(3) = \{x \in \mathbb{R} \mid f(x) = 3\}$

$f(x) = 3 \Rightarrow 3x - 5 = 3 \quad (x > 0)$ and $-3x + 1 = 3 \quad (x \leq 0)$

$3x = 8$; $-3x = 2$

$x = 8/3$ ✓ ; $x = -2/3$ ✓

$\therefore f^{-1}(3) = \left\{-\frac{2}{3}, \frac{8}{3}\right\}$.

e) $f^{-1}(-3) = \{x \in \mathbb{R} \mid f(x) = -3\}$

$f(x) = -3 \Rightarrow 3x - 5 = -3 \quad (x > 0)$ and $-3x + 1 = -3 \quad (x \leq 0)$

$\Rightarrow 3x = 2$; $-3x = -4$

$x = 2/3$ ✓ ; $x = 4/3$ ✗

$\therefore f^{-1}(-3) = \{2/3\}$

f) $f^{-1}(-6) = \{x \in \mathbb{R} \mid f(x) = -6\}$

$f(x) = -6 \Rightarrow 3x - 5 = -6 \quad (x > 0)$ and $-3x + 1 = -6 \quad (x \leq 0)$

$\Rightarrow 3x = -1$; $-3x = -7$

$x = -1/3$ ✗ ; $x = 7/3$ ✗.

$\therefore f^{-1}(-6) = \{\} = \phi$.

iii) a) $f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid f(x) \in [-5,5]\}$.
$= \{x \in \mathbb{R} \mid -5 \leq f(x) \leq 5\}$.

for $f(x) = 3x - 5 \quad (x > 0)$

$-5 \leq 3x - 5 \leq 5$.

add 5 throughout

$0 \leq 3x \leq 10$.

$\div$ by 3.

$0 \leq x \leq \frac{10}{3}$

for $f(x) = -3x + 1 \quad (x \leq 0)$

$-5 \leq -3x + 1 \leq 5$.

add -1 throughout

$-6 \leq -3x \leq 4$.

$\div$ by 3.

$-2 \leq -x \leq \frac{4}{3}$.

xply by -1

$2 \geq x \geq -4/3$.

ie $-\frac{4}{3} \leq x \leq 2$.

Combining, $f^{-1}([-5,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \leq x \leq 2 \text{ or } 0 < x \leq 10/3\}$
$= \{x \in \mathbb{R} \mid -\frac{4}{3} \leq x \leq 10/3\}$
$= [-\frac{4}{3}, \frac{10}{3}]$.

$x \in [-\frac{4}{3}, \frac{10}{3}]$.

b) $f^{-1}([-6,5]) = \{x \in \mathbb{R} \mid f(x) \in [-6,5]\}$
$= \{x \in \mathbb{R} \mid -6 \leq f(x) \leq 5\}$.

for $f(x) = 3x - 5 \quad (x > 0)$

$-6 \leq 3x - 5 \leq 5$

add 5

$-1 \leq 3x \leq 10$.

$\div$ 3

$-\frac{1}{3} \leq x \leq \frac{10}{3}$

for $f(x) = -3x + 1 \quad (x \leq 0)$

$-6 \leq -3x + 1 \leq 5$

add -1

$-7 \leq -3x \leq 4$

$\div$ 3

$-7/3 \leq -x \leq 4/3$

xply by -1.

$7/3 \geq x \geq -4/3$

ie $-\frac{4}{3} \leq x \leq 7/3$.

combining, $f^{-1}([-6,5]) = \{x \in \mathbb{R} \mid -\frac{4}{3} \leq x \leq 7/3 \text{ or } -\frac{1}{3} \leq x \leq 10/3\}$
$= \{x \in \mathbb{R} \mid -\frac{4}{3} \leq x \leq \frac{10}{3}\}$
$= [-\frac{4}{3}, \frac{10}{3}]$.

5) a) Let A and B be finite sets with $|A| = m$ and $|B| = n$. Find how many functions are possible from A to B?

b) If there are 2187 functions from A to B and $|B| = 3$, what is $|A|$?

Soln:- a) Let $A = \{a_1, a_2 \ldots a_m\}$ and $B = \{b_1, b_2 \ldots b_n\}$. Then a f^n $f: A \to B$ is of the form

$f = \{(a_1, x), (a_2, x), \ldots (a_m, x)\}$, where x stands for b_j for some j.

Since there are 'n' no. of b_j's, there are 'n' choices for x in each of the 'm' ordered pairs belonging to f.

$\therefore$ Total no. of choices for x is

$n \times n \times \ldots \times n$ (m factors) $= n^m$.

Thus there are $n^m = |B|^{|A|}$ possible f^{ns} from A to B.

b) Given $|B| = n = 3$, and $n^m = 2187$.

$\therefore$ $3^m = 2187$

$\Rightarrow m \log_e 3 = \log_e 2187$

$\Rightarrow m = |A| = 7$

5) Let $f: A \to B$ be a function. Let C and D be arbitrary nonempty subsets of B. Prove the following:

Q8

i) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

iii) $f^{-1}(\bar{C}) = \overline{f^{-1}(C)}$.

Soln:- i) for any $x \in A$,

$x \in f^{-1}(C \cup D) \Leftrightarrow f(x) \in \{C \cup D\}$

$\Leftrightarrow f(x) \in C$ or $f(x) \in D$.

$\Leftrightarrow x \in f^{-1}(C)$ or $x \in f^{-1}(D)$

$\Leftrightarrow x \in \{f^{-1}(C) \cup f^{-1}(D)\}$

$\therefore$ $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

ii) for any $x \in A$,

$$x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in \{C \cap D\}$$
$$\Leftrightarrow f(x) \in C \text{ and } f(x) \in D.$$
$$\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D)$$
$$\Leftrightarrow x \in \{f^{-1}(C) \cap f^{-1}(D)\}$$

$$\therefore f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D).$$

iii) for any $x \in A$,

$$x \in f^{-1}(\overline{C}) \Leftrightarrow f(x) \in \overline{C}$$
$$\Leftrightarrow f(x) \notin C$$
$$\Leftrightarrow x \notin f^{-1}(C).$$
$$\Leftrightarrow x \in \overline{f^{-1}(C)}.$$

$$\therefore f^{-1}(\overline{C}) = \overline{f^{-1}(C)}.$$

✓ If there are 60 1-1 fns from $A \to B$ and $|A| = 3$, then find $|B|$.

7) How many 1-1 functions are possible from $A \to B$ where $|A| = m$ & $|B| = n$, If there are 60 1-1 functions from $A \to B$ and $|A| = 3$, $|B| = ?$.

Soln :- No. of 1-1 functions from A to B is $\frac{n!}{(n-m)!}$

Given $\frac{n!}{(n-m)!} = 60$, $|A| = m = 3$, $|B| = n = ?$.

$$\frac{n!}{(n-3)!} = 60.$$

$$\Rightarrow \frac{n \times (n-1) \times (n-3)!}{(n-3)!} = 60$$

$$\Rightarrow n(n-1)(n-2) = 60.$$

$$\Rightarrow 5 \times 4 \times 3 = 60$$

$$\Rightarrow \boxed{n = 5}$$

i.e. $|B| = n = 5$.

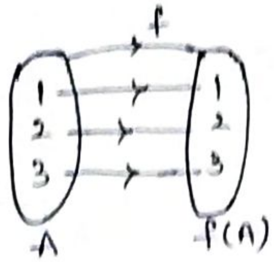
P.T.O.

8) Find the nature of the following fns defined on $A = \{1, 2, 3\}$.

i) $f = \{(1,1)\ (2,2)\ (3,3)\}$ ii) $g = \{(1,2)\ (2,2)\ (3,2)\}$.

iii) $h = \{(1,2)(2,3)(3,1)\}$.

Soln :- i) for every $a \in A$, $(a,a) \in f$ ie $a \xrightarrow{f} a$.

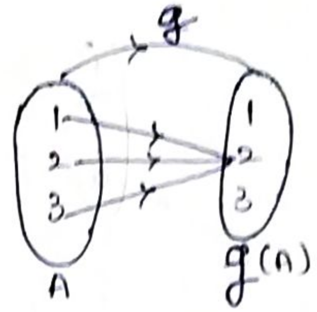


ie $A = f(A)$

ie image of every element in A is itself.

$\therefore$ f is the identity fn on A.

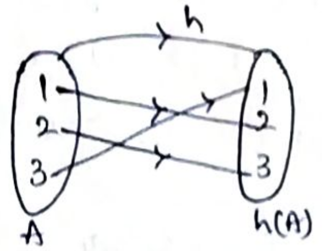
ii) we see that every $a \in A$ has 2 as its image.



ie $g(1) = 2$, $g(2) = 2$, $g(3) = 2$

$\therefore$ g is a constant fn on A.

iii)



we see that, every element of A has a unique image and every element of A has a unique preimage, under h.

$\therefore$ h is both one-to-one and onto.

$\therefore$ h is 1-1 correspondence.

9) Find whether the following fns from A to B are 1-1, onto :

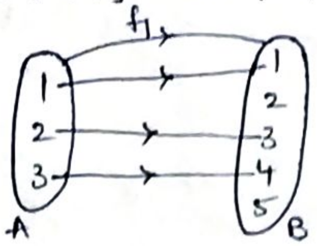
i) $f_1 = \{(1,1)\ (2,3)\ (3,4)\}$ for $A = \{1,2,3\}$, $B = \{1,2,3,4,5\}$.

ii) $f_2 = \{(1,1)\ (2,3)\ (3,4)\ (4,2)\}$ for $A = B = \{1,2,3,4\}$

iii) $A = \{a,b,c\}$ $B = \{1,2,3,4\}$, $f_3 = \{(a,1)\ (b,1)\ (c,4)\}$.

iv) $A = \{1,2,3,4\}$ $B = \{a,b,c,d\}$, $f_4 = \{(1,a)\ (2,a)\ (3,d)\ (4,c)\}$.

soln :- i)

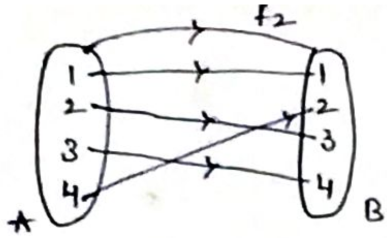


under f_1, every element of A has a unique image in B and no two elements of A have the same image in B.

$\therefore$ f_1 is one-to-one.

under f_1, the elements 2 & 5 of B has no pre-image in A. $\therefore$ f_1 is not onto.

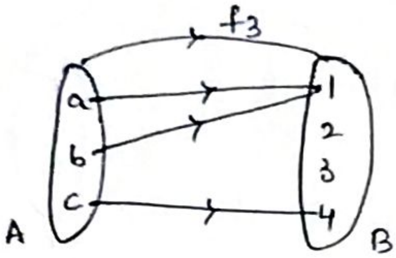
ii)



f_2 is both 1–1 & onto

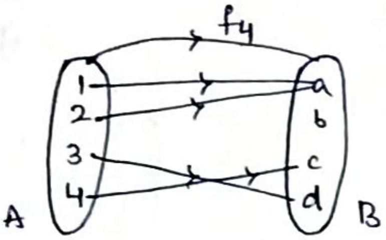
$\therefore$ f_2 is bijective.

iii)



f_3 is neither 1–1 nor onto.

iv)



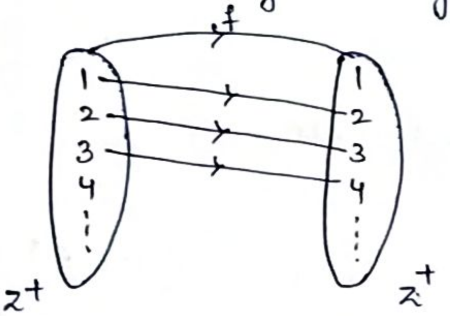
f_4 is neither 1–1 nor onto.

10) Let $f, g : Z^+ \to Z^+$ where $\forall x \in Z^+$, $f(x) = x+1$, $g(x) = \max(1, x-1)$

i) what is the range of f? ii) Is f 1–1? iii) Is f onto?

iv) what is the range of g? v) Is g 1–1? vi) Is g onto?

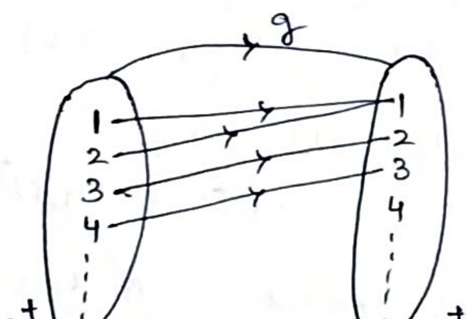
Soln :-



i) Range of $f = \{2, 3, 4 \dots\}$
$= Z^+ - \{1\}$.

ii) f is 1–1

iii) f is not onto $\because$ $f^{-1}(1)$ does not exist.



$g(x) = \max(1, x-1)$
$g(1) = \max(1, 0) = 1$.
$g(2) = \max(1, 1) = 1$
$g(3) = \max(1, 2) = 2$
$g(4) = \max(1, 3) = 3$ etc

iv) Range of $g = \{1, 2, 3 \dots\} = Z^+$.

v) g is not 1–1 $\because$ 2 elements 1, 2 are mapped to 1 under g.

vi) g is onto.

P.T.O.

11) Let $A=\{1,2,3,4\}$ and $B=\{1,2,3,4,5,6\}$.

QP i) Find how many fns are there from A to B. How many of these are 1-1? How many are onto?

ii) Find how many fns are there from B to A. How many of these are 1-1? How many are onto?

Soln :- Here $|A| = m = 4$ and $|B| = n = 6$.

i) The no. of fns possible from A to B is $n^m = 6^4 = 1296$. ($|B|^{|A|}$)

The no. of 1-1 fns possible from A to B is (for $m \le n$)

$$\frac{n!}{(n-m)!} = \frac{6!}{2!} = 360.$$

There is no onto fn from A to B. ($\because m < n$)

ii) The no. of fns possible from B to A is $m^n = 4^6 = 4096$. ($|A|^{|B|}$)

There is no 1-1 fn from B to A ($\because m < n$).

The no. of onto fns from B to A is

$$p(n,m) = \sum_{k=0}^{m} (-1)^k \left({}^mC_{m-k}\right)(m-k)^n$$

$$p(6,4) = \sum_{k=0}^{4} (-1)^k \left({}^4C_{4-k}\right)(4-k)^6$$

$$= {}^4C_4 \times 4^6 - {}^4C_3 \times 3^6 + {}^4C_2 \times 2^6 - {}^4C_1 \times 1^6$$

$$= 1560.$$

Note :- 1) 1-1 $\rightarrow \frac{n!}{(n-m)!}$ for $m \le n$

2) onto $\rightarrow p(m,n)$ for $m \ge n$.

12) There are 6 Programmers who can assist eight executives. In how many ways can the executives be assisted so that each programmer assists atleast one executive?

use fn defn concept no 'a' in A shud be free

Soln :- all programmer shud have an executive.

Let A denote the set of executives and B denote the set of programmers. $\therefore |A| = m = 8$, $|B| = n = 6$.

P.T.O.

Thus Required no. = no. of onto fns from A to B.

$\Rightarrow p(m,n) = p(8,6) = (6!) \times S(8,6)$.

$\therefore S(8,6) = \frac{1}{6!} \sum_{k=0}^{6} (-1)^k \left({}^6C_{(6-k)}\right) (6-k)^8$

$= \frac{1}{6!} \{ {}^6C_6 \times 6^8 - {}^6C_5 \times 5^8 + {}^6C_4 \times 4^8 - {}^6C_3 \times 3^8 + {}^6C_2 \times 2^8 - {}^6C_1 \times 1^8 \}$

$= 266$.

$\therefore p(8,6) = 6! \times 266 = 191520$.

13) Find the no. of ways of distributing 4 distinct objects among 3 identical containers, with some container(s) possible empty.

Soln:- Here, no. of objects is $m=4$ and no. of containers is $n=3$. (QP $m=6$, $n=4$)

$\therefore$ Req. no. $= p(m) = \sum_{i=1}^{n} S(m,i)$

$p(4) = \sum_{i=1}^{3} S(4,i) = S(4,1) + S(4,2) + S(4,3)$ $\rightarrow$ (1)

$S(4,1) = \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \left({}^1C_{1-k}\right) (1-k)^4 = 1$.

$S(4,2) = \frac{1}{2!} \sum_{k=0}^{2} (-1)^k \left({}^2C_{2-k}\right) (2-k)^4 = \frac{1}{2}\left[2^4 - {}^2C_1 \times 1^4\right] = 7$.

|||ly $S(4,3) = 6$.

$\therefore$ (1) $\Rightarrow$ Req. no. $= 1+7+6 = 14$.

14) Find the no. of equivalence relations that can be defined on a finite set A with $|A| = 6$. QP

Soln:- Since $|A| = m = 6$, a partition of A can have atmost 6 cells. Treating the elements of A as objects & cells as containers, the no. of partitions having k cells is $S(6,k)$. Since k varies from 1 to 6, the total no. of different partitions of A is

$$p(m) = \sum_{i=1}^{n} S(m,i).$$

$\therefore p(6) = \sum_{i=1}^{6} S(6,i)$

$p(6) = S(6,1) + S(6,2) + S(6,3) + S(6,4) + S(6,5) + S(6,6)$ $\hookrightarrow$ (1)

we know that $S(m,n) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k ({}^nC_{n-k}) (n-k)^m$.

$S(6,1) = \frac{1}{1!} \sum_{k=0}^{1} (-1)^k \, {}^1C_{1-k} \, (1-k)^6 = 1$.

$S(6,2) = \frac{1}{2!} \sum_{k=0}^{2} (-1)^k \, {}^2C_{2-k} \, (2-k)^6 = \frac{1}{2}\left[{}^2C_2 \times 2^6 - {}^2C_1 \times 1^6\right] = 31$

$S(6,3) = \frac{1}{3!} \sum_{k=0}^{3} (-1)^k \, {}^3C_{3-k} \, (3-k)^6 = \frac{1}{6}\left[{}^3C_3 \times 3^6 - {}^3C_2 \times 2^6 + {}^3C_1 \times 1^6\right] = 90$

$S(6,4) = 65$, $S(6,5) = 15$, $S(6,6) = 1$.

$\therefore$ No. of partitions of A is

(1) $\Rightarrow$ $p(6) = 203$.

Since each partition of A corresponds to an equivalence relⁿ on A, it follows that if $|A| = 6$, then 203 equivalence relations can be defined on A.

15) Let $A = \mathbb{R}$, $B = \{x \mid x$ is real and $x \geq 0\}$. Is the fⁿ $f: A \to B$ defined by $f(a) = a^2$ an onto fⁿ? Is it a 1-1 fⁿ?.

QP

Solⁿ:- i) Take any $b \in B$, then b is non-negative real no.

Since $f(a) = a^2 \Rightarrow b = a^2 \Rightarrow a = \pm\sqrt{b} \in A$ $(\because A = \mathbb{R})$.

we ~~see that~~, ~~since $f(a) = a^2$,~~

$\cancel{f(\sqrt{b}) = (\sqrt{b})^2 = b}$.

$\cancel{f(-\sqrt{b}) = (-\sqrt{b})^2 = b}$.

Thus $\pm\sqrt{b}$ are the preimages of b under f.

Since 'b' is arbitrary element of B, every element in B has a preimage in A.

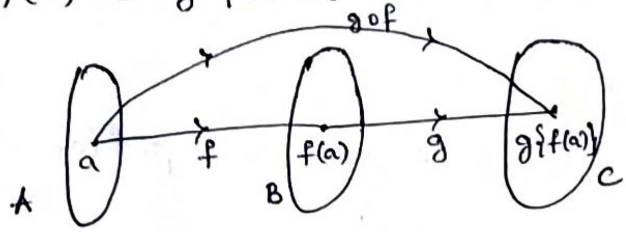
$\therefore$ f is onto fⁿ.

ii) Since $b \in B$ has 2 preimages $\pm\sqrt{b} \in A$ under f, f is not one-to-one.

Composition of functions :-

Consider 3 non-empty sets A, B, C and the fns $f : A \to B$ and $g : B \to C$. The composition (or product) of these 2 functions is defined as $g \circ f : A \to C$ with

$$(g \circ f)(a) = g\{f(a)\} \quad \forall\ a \in A.$$



Note :- 1) For $f : A \to A$, $f^1 = f$, $f^2 = f \circ f$, $f^3 = f \circ f^2$ & so on.

Thus $f^1 = f$, $f^n = f \circ f^{n-1}$.

→ done in Problem(2)

2) Let $f : A \to B$ and $g : B \to C$, then

i) If f and g are one-to-one, then $g \circ f$ is also one-to-one

ii) If $g \circ f$ is one-to-one, then f is one-to-one.

iii) If f and g are onto, then $g \circ f$ is also onto.

iv) If $g \circ f$ is onto, then g is onto.

3) The composition of 2 fns is not commutative i.e $g \circ f \neq f \circ g$.

Problems :-

1) Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be 3 functions.

Then $(h \circ g) \circ f = h \circ (g \circ f)$.

Soln :- Since $f : A \to B$, $g : B \to C$ and $h : C \to D$, we have both $(h \circ g) \circ f$ and $h \circ (g \circ f)$ are fns from A to D.

For any $x \in A$,

$$[(h \circ g) \circ f](x) = (h \circ g)[f(x)] = (h \circ g)(y) \quad \text{where } y = f(x)$$
$$= h[g(y)] = h(z) \quad \text{where } z = g(y). \to ①$$

$$[h \circ (g \circ f)](x) = h[(g \circ f)(x)]$$
$$= h[g\{f(x)\}]$$
$$= h[g(y)] = h(z) \to ②$$

Thus $[(h \circ g) \circ f](x) = [h \circ (g \circ f)](x)$ for every $x \in A$.

$\therefore (h \circ g) \circ f = h \circ (g \circ f) \Rightarrow$ Compositions of 3 fns is associative.

2) Let $f: A \to B$ and $g: B \to C$, Prove that

i) If f and g are 1-1, then $g \circ f$ is 1-1.

ii) If $g \circ f$ is 1-1, then f is 1-1.

iii) If f and g are onto, then $g \circ f$ is onto.

iv) If $g \circ f$ is onto, then g is onto.

Sol^n:- Let $a_1, a_2 \in A$.

i) Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$.

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow g(b_1) = g(b_2)$ [$\because f: A \to B \Rightarrow b_1 = f(a_1)$ & $b_2 = f(a_2)$]

$\Rightarrow b_1 = b_2$ ($\because g$ is 1-1)

$\Rightarrow f(a_1) = f(a_2)$

$\Rightarrow a_1 = a_2$ ($\because f$ is 1-1)

Thus if $(g \circ f)(a_1) = (g \circ f)(a_2)$, then $a_1 = a_2$.

$\Rightarrow g \circ f$ is 1-1.

ii) Let $f(a_1) = f(a_2)$.

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow (g \circ f)(a_1) = (g \circ f)(a_2)$

$\Rightarrow a_1 = a_2$ ($\because g \circ f$ is 1-1)

Thus if $f(a_1) = f(a_2)$, then $a_1 = a_2$

$\Rightarrow f$ is 1-1.

iii) f is onto $\Rightarrow \forall b \in B, \exists a \in A$ such that $b = f(a) \to$ ①

g is onto $\Rightarrow \forall c \in C, \exists b \in B$ such that $c = g(b) \to$ ②

from ②, $c = g(b) = g(f(a))$ (using ①)

$c = (g \circ f)(a)$

$\therefore \forall c \in C, \exists a \in A$ such that $c = (g \circ f)(a) \Rightarrow g \circ f$ is onto.

iv) $g \circ f$ is onto $\Rightarrow \forall c \in C, \exists a \in A$ such that $c = (g \circ f)(a)$.

$\Rightarrow c = g(f(a))$

$= g(b)$, $b = f(a) \in B$.

$\therefore \forall c \in C, \exists b = f(a) \in B$ such that $c = g(b)$

$\therefore g$ is onto.

3) Let $A = \{1,2,3,4\}$ and $f: A \to A$ is a fn defined by $f = \{(1,2)(2,2)(3,1)(4,3)\}$. find f^2.

Soln :- Given : $f(1) = 2$, $f(2) = 2$, $f(3) = 1$, $f(4) = 3$.

$\therefore$ $f^2(1) = f \circ f(1) = f(f(1)) = f(2) = 2$.

$f^2(2) = f \circ f(2) = f(f(2)) = f(2) = 2$.

$f^2(3) = f \circ f(3) = f(f(3)) = f(1) = 2$

$f^2(4) = f \circ f(4) = f(f(4)) = f(3) = 1$.

i.e. $f^2 = \{(1,2)(2,2)(3,2)(4,1)\}$.

4) Consider the fns f and g defined by $f(x) = x^3$ and $g(x) = x^2+1$, $\forall x \in \mathbb{R}$, find $g \circ f$, $f \circ g$, f^2 and g^2.

Soln :- $(g \circ f)(x) = g\{f(x)\} = g(x^3) = (x^3)^2+1 = x^6+1$.

$(f \circ g)(x) = f\{g(x)\} = f(x^2+1) = (x^2+1)^3 = (x^2)^3+1^3+3.x^2.1(x^2+1)$

$= x^6+1+3x^4+3x^2$

$f^2(x) = (f \circ f)(x) = f\{f(x)\} = f(x^3) = (x^3)^3 = x^9$

$g^2(x) = (g \circ g)(x) = g\{g(x)\} = g(x^2+1) = (x^2+1)^2+1 = x^4+2+2x^2$.

5) Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$ and $C = \{w,x,y,z\}$ with $f: A \to B$ and $g: B \to C$ given by $f = \{(1,a)(2,a)(3,b)(4,c)\}$ and $g = \{(a,x)(b,y)(c,z)\}$. Find $g \circ f$.

Soln :- By data, $f(1) = a$ and $g(a) = x$

$f(2) = a$ $\quad g(b) = y$

$f(3) = b$ $\quad g(c) = z$.

$f(4) = c$

Since $f: A \to B$ and $g: B \to C$, $g \circ f: A \to C$.

$\therefore$ $g \circ f(1) = g(f(1)) = g(a) = x$.

$g \circ f(2) = g(f(2)) = g(a) = x$.

$g \circ f(3) = g(f(3)) = g(b) = y$.

$g \circ f(4) = g(f(4)) = g(c) = z$.

Thus $g \circ f = \{(1,x)(2,x)(3,y)(4,z)\}$.

P.T.O.

6) Let f, g, h be fns from $\mathbb{Z}$ to $\mathbb{Z}$ defined by

QP $f(x) = x-1$, $g(x) = 3x$, $h(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd}. \end{cases}$

Determine $(f\circ(g\circ h))(x)$ and $((f\circ g)\circ h)(x)$ and verify that $f\circ(g\circ h) = (f\circ g)\circ h$.

Soln :- we have $(g\circ h)(x) = g\{h(x)\} = 3h(x)$.

$$\therefore (f\circ(g\circ h))(x) = f\{(g\circ h)(x)\}.$$
$$= f\{g\{h(x)\}\}.$$
$$= f\{3h(x)\} = 3h(x) - 1.$$
$$= \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd}. \end{cases} \rightarrow (1)$$

$$\cancel{((f\circ g)\circ h)(x) = (f\circ g)\{h(x)\}} \quad \cancel{= f\{g\{h(x)}$$

we have $(f\circ g)(x) = f\{g(x)\} = g(x) - 1 = 3x - 1$.

$$\therefore ((f\circ g)\circ h)(x) = (f\circ g)\{h(x)\}$$
$$= 3h(x) - 1 = \begin{cases} -1 & \text{if } x \text{ is even} \\ 2 & \text{if } x \text{ is odd}. \end{cases} \rightarrow (2)$$

from (1) & (2),

$$f\circ(g\circ h) = (f\circ g)\circ h.$$

7) Let f and g be functions from $\mathbb{R}$ to $\mathbb{R}$ defined by

QP $f(x) = ax+b$ and $g(x) = 1 - x + x^2$. If $(g\circ f)(x) = 9x^2 - 9x + 3$

Determine a & b.

Soln:. $(g\circ f)(x) = 9x^2 - 9x + 3$.

$$g\{f(x)\} = 9x^2 - 9x + 3$$
$$\Rightarrow g(ax+b) = 9x^2 - 9x + 3$$
$$\Rightarrow 1 - (ax+b) + (ax+b)^2 = 9x^2 - 9x + 3$$
$$\Rightarrow 1 - (ax+b) + (a^2x^2 + b^2 + 2abx) = 9x^2 - 9x + 3$$
$$\Rightarrow (1 - b + b^2) + (2ab - a)x + a^2x^2 = 9x^2 - 9x + 3$$

∴ we have $a^2 = 9$.

$$2ab - a = -9$$

$$1 - b + b^2 = 3.$$

∴ $a^2 = 9 \Rightarrow \boxed{a = \pm 3}$

Now, $2ab - a = -9 \Rightarrow$

when $a = 3$; $6b - 3 = -9$

$6b = -6 \Rightarrow \boxed{b = -1}$

when $a = -3$; $-6b + 3 = -9$

$-6b = -12$

$\boxed{b = 2}$

∴ $a=3$, $b=-1$ and $a=-3$, $b=2$ are the required values.

8) Let $f, g, h : R \to R$ where $f(x) = x^2$, $g(x) = x+5$,
QP $h(x) = \sqrt{x^2+2}$, S.T $(hog) \circ f = h \circ (g \circ f)$.

Soln:- consider $(h \circ g)(x) = h\{g(x)\}$

$$= h\{x+5\}$$

$$= \sqrt{(x+5)^2+2}$$

$$= \sqrt{x^2+25+10x+2}$$

$$= \sqrt{x^2+10x+27}.$$

Now LHS: $((hog) \circ f)(x) = (h \circ g)\{f(x)\}$

$$= \sqrt{(f(x))^2 + 10f(x) + 27}$$

$$= \sqrt{x^4 + 10x^2 + 27}.$$

consider $(g \circ f)(x) = g\{f(x)\} = g\{x^2\} = x^2+5$.

Now RHS: $(h \circ (g \circ f))(x) = h\{(g \circ f)(x)\}$

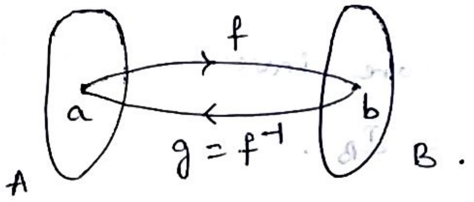
$$= h(x^2+5)$$

$$= \sqrt{(x^2+5)^2+2} = \sqrt{x^4+25+10x^2+2} = \sqrt{x^4+10x^2+27}$$

∴ LHS = RHS.

Invertible functions :- A fn $f: A \to B$ is said to be invertible if there exists a fn $g: B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$, where I_A is the Identity fn on A and I_B is the Identity fn on B. Then g is called an inverse of f and we write $g = f^{-1}$.

[since $f: A \to B$, $g: B \to A$
$g \circ f : A \to A$
$f \circ g : B \to B$



Note :-

1) If a fn $f: A \to B$ is invertible then it has a unique inverse. further, if $f(a) = b$ then $f^{-1}(b) = a$.
2) If f is invertible, then $f(a) = b$ and $a = f^{-1}(b)$ are equivalent.
3) If $f = \{(a,b) \mid a \in A, b \in B\}$ is invertible, then
   $f^{-1} = \{(b,a) \mid b \in B, a \in A\}$ and conversely.
4) If f is invertible then f^{-1} is invertible and $(f^{-1})^{-1} = f$.
5) A fn $f: A \to B$ is invertible iff it is one-to-one and onto.
6) If $f: A \to B$ and $g: B \to C$ are invertible fns then $g \circ f : A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
7) Let A and B be finite sets with $|A| = |B|$ and f be a fn from A to B. Then the following statements are equivalent
   i) f is one-to-one.
   ii) f is onto.
   iii) f is invertible.

P.T.O

Problems :-

1) Suppose $f : A \to B$ is invertible, then P.T

Prove that f has unique inverse. further if $f(a) = b$ then $a = f^{-1}(b)$.

Soln :- Given f is invertible

Let g be inverse of f. ie $g = f^{-1}$.

$\therefore \quad g \circ f = I_A$ and $f \circ g = I_B$.

Let us assume there is one more inverse for f, say h.

$\therefore \quad h \circ f = I_A$ and $f \circ h = I_B$.

consider $h \circ (f \circ g) = (h \circ f) \circ g$.

$h \circ I_B = I_A \circ g$.

$\therefore \quad h = g$ #

$\therefore$ f has unique inverse.

Further, $f(a) = b$.

$\Rightarrow g(f(a)) = g(b)$

$\Rightarrow g \circ f(a) = g(b)$

$\Rightarrow I_A(a) = g(b)$

$\Rightarrow a = g(b)$

$\Rightarrow a = f^{-1}(b) \quad (\because g = f^{-1})$

2) A f^n $f : A \to B$ is invertible iff it is one-to-one and onto.

QP

Soln :- Let $f : A \to B$ be a invertible f^n, then there exists a unique f^n $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$.

TPT f is one-to-one

Let $f(a_1) = f(a_2)$

$\Rightarrow g(f(a_1)) = g(f(a_2))$

$\Rightarrow g \circ f(a_1) = g \circ f(a_2)$

$\Rightarrow I_A(a_1) = I_A(a_2)$

$\Rightarrow a_1 = a_2$

$\therefore$ f is 1-1.

T f is onto.

Take any $b \in B$, then $g(b) \in A$.

and $b = I_B(b) = (f \circ g)(b) = f\{g(b)\}$

Thus b is the image of an element $g(b) \in A$ under f.

$\therefore$ f is onto.

Conversely, Suppose f is 1-1 and onto. $\therefore$ It is bijective.

$\therefore$ for every $b \in B$ there exists a unique $a \in A$ such that

$f(a) = b$.

Consider a f^n $g: B \to A$ defined by $g(b) = a$.

then $(f \circ g)(b) = f\{g(b)\} = f(a) = b = I_B(b)$

$(g \circ f)(a) = g\{f(a)\} = g(b) = a = I_A(a)$.

$\therefore$ f is Invertible with g as the Inverse.

3) If $f: A \to B$ and $g: B \to C$ are invertible f^{ns}, then Prove that $g \circ f: A \to C$ is also invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

QP

Sol^n:- Suppose f and g are invertible.

$\therefore$ f and g are 1-1 and onto.

$\therefore$ $g \circ f$ is also 1-1 and onto.

Hence $g \circ f$ is Invertible.

Next, we have $f: A \to B$, $g: B \to C$, $g \circ f: A \to C$.

$\therefore$ $f^{-1}: B \to A$, $g^{-1}: C \to B$, and let $h = f^{-1} \circ g^{-1}: C \to A$.

consider $h \circ (g \circ f) = (f^{-1} \circ g^{-1}) \circ (g \circ f)$

$= f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ I_B \circ f = f^{-1} \circ f = I_A \to ①$

consider $(g \circ f) \circ h = (g \circ f) \circ (f^{-1} \circ g^{-1})$

$= g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ I_A \circ g^{-1} = g \circ g^{-1} = I_B \to ②$

from ① & ②, h is the inverse of $g \circ f$.

i.e. $h = (g \circ f)^{-1}$.

Thus $(g \circ f)^{-1} = h = f^{-1} \circ g^{-1}$.

4) Let $A = \{1,2,3,4\}$ and f and g be fns from A to A

given by $f = \{(1,4)(2,1)(3,2)(4,3)\}$ and $g = \{(1,2)(2,3)(3,4)(4,1)\}$.

Prove that f and g are inverses of each other.

Soln :- By data, $f(1) = 4,\ f(2) = 1,\ f(3) = 2,\ f(4) = 3$

$g(1) = 2,\ g(2) = 3,\ g(3) = 4,\ g(4) = 1$.

we have
$$(g\circ f)(1) = g\{f(1)\} = g(4) = 1 = I_A(1).$$
$$(g\circ f)(2) = g\{f(2)\} = g(1) = 2 = I_A(2)$$
$$(g\circ f)(3) = g\{f(3)\} = g(2) = 3 = I_A(3).$$
$$(g\circ f)(4) = g\{f(4)\} = g(3) = 4 = I_A(4).$$

and
$$(f\circ g)(1) = f\{g(1)\} = f(2) = 1 = I_A(1).$$
$$(f\circ g)(2) = f\{g(2)\} = f(3) = 2 = I_A(2).$$
$$(f\circ g)(3) = f\{g(3)\} = f(4) = 3 = I_A(3).$$
$$(f\circ g)(4) = f\{g(4)\} = f(1) = 4 = I_A(4).$$

Thus $\forall\ x \in A$, $(g\circ f)(x) = I_A(x)$ and $(f\circ g)(x) = I_A(x)$.

$\Rightarrow$ g is an inverse of f and f is an Inverse of g.

i.e. f and g are inverses of eachother.

6) Let $A = B = \mathbb{R}$, the set of all real no.'s, and the fns

QP $f : A \to B$ and $g : B \to A$ be defined by

$f(x) = 2x^3 - 1, \ \forall \ x \in A \ ; \ g(y) = \left\{\frac{1}{2}(y+1)\right\}^{1/3}, \ \forall \ y \in B.$

Show that each of f and g is the inverse of the other.

Soln :- for any $x \in A$,

$$(g \circ f)(x) = g\{f(x)\} = g(y) \quad \text{where } y = f(x)$$

$$= \left\{\frac{1}{2}(y+1)\right\}^{1/3}$$

$$= \left\{\frac{1}{2}(2x^3 - 1 + 1)\right\}^{1/3} \quad (\because \ y = f(x) = 2x^3 - 1)$$

$$= x.$$

$\therefore \ g \circ f = I_A$.

for any $y \in B$,

$$(f \circ g)(y) = f\{g(y)\}$$

$$= f(z) \quad \text{where } z = g(y).$$

$$= 2z^3 - 1.$$

$$= 2(g(y))^3 - 1 = 2\left(\left\{\frac{1}{2}(y+1)\right\}^{1/3}\right)^3 - 1.$$

$$= 2\left\{\frac{1}{2}(y+1)\right\} - 1 = y + 1 - 1 = y.$$

$\therefore \ f \circ g = I_B$.

$\therefore$ each of f and g is an invertible fn.

$\Rightarrow$ f and g are inverses of each other.

7) Consider the fn $f : R \to R$ defined by $f(x) = 2x + 5$. Let

QP a fn $g : R \to R$ be defined by $g(x) = \frac{1}{2}(x - 5)$. Prove

that g is an inverse of f.

P.T.O.

Sol^n:- for any $x \in R$,

$$(g \circ f)(x) = g\{f(x)\} = g(2x+5)$$
$$= \frac{1}{2}(2x+5-5) = \frac{1}{2}(2x) = x = I_R(x).$$

$$(f \circ g)(x) = f\{g(x)\} = f\left\{\frac{1}{2}(x-5)\right\} = 2 \times \frac{1}{2}(x-5)+5$$
$$= x-5+5 = x = I_R(x).$$

$\therefore$ g is an inverse of f. (Also f is an inverse of g).

5) Let $A = B = C = R$ and $f: A \to B$ and $g: B \to C$ be defined by $f(a) = 2a+1$ and $g(b) = \frac{1}{3}b$, $\forall\, a \in A$, $\forall\, b \in B$. compute $g \circ f$ and S.T $g \circ f$ is invertible. Also find $(g \circ f)^{-1}$

QP

Sol^n:- we have $g \circ f : A \to C$

$$\therefore (g \circ f)(a) = g\{f(a)\} = g(2a+1) = \frac{1}{3}(2a+1).$$

To Prove: $g \circ f$ is invertible, we prove $f: A \to B$ and $g: B \to C$ are invertible; ie we prove f & g are 1-1 and onto.

To Prove: f is 1-1

consider $f(a_1) = f(a_2)$
$\Rightarrow 2a_1 + 1 = 2a_2 + 1$
$\Rightarrow 2a_1 = 2a_2$
$\Rightarrow a_1 = a_2$
$\Rightarrow f$ is 1-1

To Prove: f is onto.

$\forall\, b \in B$, then $b = f(a) = 2a+1$
$\Rightarrow a = f^{-1}(b) = \frac{b-1}{2}$

ie for every $b \in B$, $\exists$ a preimage $a = \frac{b-1}{2}$ in A. $\Rightarrow f$ is onto.

$\therefore f$ is invertible.

To Prove: g is 1-1.

consider $g(b_1) = g(b_2) \Rightarrow \frac{b_1}{3} = \frac{b_2}{3} \Rightarrow b_1 = b_2$

$\Rightarrow g$ is 1-1.

To Prove: g is onto.

$\forall\ c \in C$, we have $c = g(b) = \frac{b}{3}$.

$\Rightarrow b = 3c$.

ie for every $c \in C$, $\exists$ a preimage $b = g^{-1}(c) = 3c$ in B.

$\therefore$ g is onto.

Thus g is invertible.

Since f & g are invertible, we have $g \circ f$ is also invertible.

Now, $(g \circ f)^{-1}(c) = (f^{-1} \circ g^{-1})(c)$ [since $g \circ f : A \to C$, $(g \circ f)^{-1} : C \to A$]

$= f^{-1}\{g^{-1}(c)\}$.

$= f^{-1}(3c)$

$= \dfrac{3c-1}{2}$.

THE PIGEONHOLE PRINCIPLE

Statement :- If 'm' pigeons occupy 'n' pigeonholes and if $m > n$, then two or more pigeons occupy the same pigeonhole.

QP

(OR)

If 'm' pigeons occupy 'n' pigeonholes where $m > n$, then atleast one pigeonhole must contain two or more pigeons in it.

Ex :- If 8 children are born in the same week, then two or more children are born on the same day of the week.

Generalised pigeon hole Principle :-

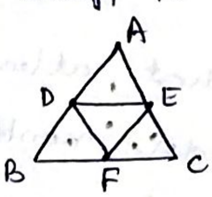
If 'm' pigeons occupy 'n' pigeonholes and $m > n$, then atleast one pigeonhole must contain $p+1$ or more pigeons in it, where $p = \left\lfloor \frac{(m-1)}{n} \right\rfloor$

Problems :-

1) ABC is an equilateral Δ^{le} whose sides are of length 1 cm each. If we select 5 points inside the Δ^{le}, prove that atleast 2 of these points are such that the distance b/w them is less than $1/2$ cm.

QP

Soln :-



Consider the equilateral Δ^{le} ABC whose sides are of length 1 cm each. Consider the Δ^{le} DEF formed by the midpoints of the sides AB, AC and BC resp (see fig). This divides Δ^{le} ABC into 4 small equilateral Δ^{les}, whose length of each side is $1/2$ cm.

Let us treat 5 points as pigeons and 4 small equilateral Δ^{les} as pigeon holes, then by pigeon hole principle, atleast one small Δ^{le} contains 2 or more points and the distance b/w such points is less than $1/2$ cm.

2) A bag contains 12 pair of socks (each pair in different color).
If a person draws the socks one by one at random,
determine at most how many draws are required to get
atleast one pair of matched socks.

Soln :- Let n be the no. of draws.
For $n \leq 12$, it is possible that the socks drawn are of different
colors ∵ there are 12 colors.
For $n = 13$, all socks cannot have different colors - atleast two
must have same color.
Let us treat 13 as the no. of pigeons and 12 colors as 12 pigeonholes.
∴ atmost 13 draws are required to have atleast 1 pair of
socks of the same color.

3) If 5 colors are used to paint 26 doors, prove that atleast
6 doors will have same color.

Soln :- consider 26 doors as pigeons and 5 colors as pigeon
holes. By generalised pigeon hole principle, atleast 1 color
must be assigned to $p+1$ or more doors.

ie $p+1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{26-1}{5} \right\rfloor + 1 = 5+1 = 6.$

4) How many persons must be chosen in order that atleast five
of them will have birthdays in the same calender month?

Soln :- Let 'm' be the no. of persons. Number of months
over which the birthdays are distributed is $n = 12$.
The least no. of persons having birthday in the same month
is $5 = p+1$.

ie $p+1 = 5 \Rightarrow \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 5 \Rightarrow \left\lfloor \frac{m-1}{12} \right\rfloor = 4$

$\Rightarrow m-1 = 48 \Rightarrow \boxed{m = 49}$

$\therefore$ No. of persons is 49 (at the least).

5) Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then atleast one of the dictionaries must have atleast 2045 pages.

Q

Soln: Consider 61,327 pages as pigeons i.e $m = 61,327$ and 30 dictionaries as pigeon holes i.e $n = 30$.

By using the generalised pigeonhole principle, atleast 1 dictionary must contain $p+1$ or more pages.

i.e $p+1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{61,327-1}{30} \right\rfloor + 1 = \lfloor 2044.2 \rfloor + 1$

$= 2044 + 1$

$= 2045$.

This proves the required result.

6) If any $n+1$ numbers are chosen from 1 to $2n$, then

Q Show that atleast one pair add to $2n+1$.

Soln:- Let us consider the following sets:

$A_1 = \{1, 2n\}$, $A_2 = \{2, 2n-1\}$, $A_3 = \{3, 2n-2\}$ .....

$A_{n-1} = \{n-1, n+2\}$, $A_n = \{n, n+1\}$.

These are the only sets containing 2 no's from 1 to $2n$ whose sum is $2n+1$.

Since there are only n sets (pigeonholes), two of the $n+1$ chosen numbers (pigeons) belong to same set (by pigeon hole principle) and sum of these 2 no's $= 2n+1$.

7) Prove that if 101 integers are selected from the set $S = \{1, 2, 3, \ldots 200\}$, then atleast 2 of these are such that one divides the other.

Soln:- we have $S = \{1, 2, 3, \ldots 200\}$.

Let $X = \{1, 3, 5 \ldots 199\}$. $\Rightarrow$ $|X| = 100$.

Any element n in the set S can be written as

$n = 2^k \times x$ where k is an integer ≥ 0 and $x \in X$.

Ex:- $1 = 2^0 \times 1$.

$2 = 2^1 \times 1$

$3 = 2^0 \times 3$

$4 = 2^2 \times 1$ etc

Considering 101 elements as pigeons and elements in X as pigeonhole, by pigeonhole principle, atleast 2 elements out of 101 must be selected to same $x \in X$.

Let a and b be such elements, then $a = 2^{k_1} x$, $b = 2^{k_2} x$, where k_1 and k_2 are integers ≥ 0.

Thus if $k_1 \leq k_2$ then a divides b.

if $k_1 > k_2$ then b divides a.
(ie $k_2 < k_1$)

8) Shirts numbered consecutively from 1 to 20 are worn by 20 students of a class. when any 3 of these students are chosen to be a debating team from the class, the sum of their shirt numbers is used as the code no. of the team. Show that if any 8 of the 20 are selected, then from these 8 we may form atleast 2 different teams having the same code no.

Sol^n:- From the 8 of the 20 students selected, the no. of teams of 3 students that can be formed is $^8C_3 = 56$.

Smallest possible code no. is $1+2+3 = 6$.

Largest " ———————— " $18+19+20 = 57$.

$\therefore$ code no.'s vary from 6 to 57 (inclusive) and these are 52 in no.

Let us consider no. of teams as pigeons i.e. $m = 56$.

and the no. of codes as pigeon holes i.e. $n = 52$.

$\therefore$ By pigeon hole principle, atleast 2 different teams will have the same code no, i.e. atleast 1 code must be assigned to 2 or more teams.

9) S.T in any set of 29 persons, atleast 5 persons have been born on the same day of the week.

QP

Sol^n:- Since there are 7 days in a week,

let the no. of persons born be the pigeons i.e $m = 29$.

and the no. of days of a week be pigeon holes. i.e $n = 7$.

then by pigeon hole principle, atleast 1 pigeonhole must contain $p+1$ or more pigeons in it.

i.e atleast 1 day must contain $p+1$ or more person's born day.

i.e $p+1 = \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{29-1}{7} \right\rfloor + 1 = \frac{28}{7} + 1 = 4+1 = 5$.

$\therefore$ atleast 5 persons have been born on the same day of the week.

(OR) Treating the 7 days of a week as 7 pigeonholes and 29 persons as 29 pigeons. By generalised pigeonhole principle, atleast one day of the week is assigned to p+1 or more persons.

$\Rightarrow \left\lfloor \frac{m-1}{n} \right\rfloor + 1 = \left\lfloor \frac{29-1}{7} \right\rfloor + 1 = 5$ or more persons.

∴ atleast 5 of any 29 persons must have been born on the same day of the week.

10) S.T if any 7 numbers from 1 to 12 are chosen, then 2 of them will add to 13.

QP

Soln:- Let us consider the following sets:

$A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$

$A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$.

These are the only sets containing 2 no's from 1 to 12 whose sum is 13.

Since there are only 6 sets (6 pigeon holes), 2 of the 7 chosen numbers (7 pigeons) belong to same set (by pigeon hole principle) and sum of these 2 no's = 13.

MATHEMATICAL FOUNDATIONS FOR COMPUTING, PROBABILITY & STATISTICS

21MATCS41

Module – II : Introduction to Graph Theory

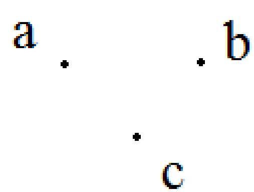
Definitions & Examples

Graph: A ***graph*** is a pair (V, E) where V is a non-empty set and E is a set of unordered pairs of elements taken from the set V.

- For a graph (V, E) the elements of V are called **vertices** or **points** or **nodes** and the elements of E are called **edges** or undirected edges. The set V is called the **vertex set** and the set E is called the **edge set**.
- The graph (V, E) is also denoted by G = (V, E) or G = G (V, E) or G.

Null graph: A graph containing no edge is called a ***null graph***.

Ex:



Here V = {a, b, c} and E = { }

Trivial graph: A null graph with only one vertex is called a ***trivial graph***.

Ex:

Here V = {a} and E = { }

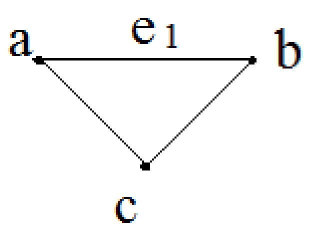
Finite graph: A graph with only a finite number of vertices and edges is called a ***finite graph*** otherwise it is called an ***infinite graph***.

Order and Size: The number of vertices in a graph is called the ***order of the graph*** **and** the number of edges in it is called its ***size.***

In other words, for a graph G = (V, E) the cardinality of the set V denoted by |V|, is called the **order** of G and the cardinality of the set E denoted by |E|, is called the **size** of G. ***A graph of order n and size m is called a (n, m) graph***.

End Vertices: If v_i and v_j denote two vertices of a graph and if e_k denotes the edge joining v_i and v_j, then v_i and v_j are called the **end vertices** of e_k .

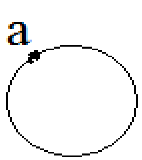
Ex:



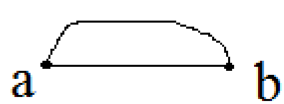
Here a and b are the end vertices of e_1.

Loop: An edge whose end vertices are one and the same is known as a ***loop***.

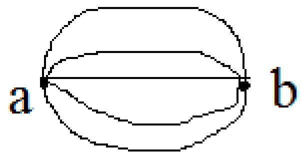
Ex:

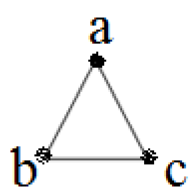


Parallel edges: Two edges which have the same end vertices are known as ***parallel edges.*** Ex:

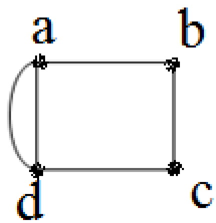


Multiple edges: If in a graph, there are two or more edges with the same end vertices, the edges are called **multiple edges.**

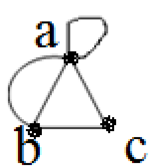
Ex: 

Simple graph: A graph which does not contain loops and multiple edges is called a ***simple graph***. Ex: 

Multi graph: A graph which contains multiple edges but no loops is called a ***multi graph***.

Ex: 

General graph: A graph which contains multiple edges and / or loops is called a ***general graph***



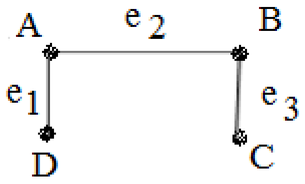
Incidence: When a vertex *v* of a graph *G* is an end vertex of an edge *e* in a graph G, then the edge *e is* ***incident on*** or ***to*** the vertex *v.*

- Since every edge has two end vertices, every edge is incident on two vertices, one at each end.
- The two end vertices are ***co-incident*** if the edge is a loop.

Adjacent vertices: Two ***vertices*** are said to be ***adjacent vertices*** or ***neighbors*** if there is an edge joining them.

Adjacent edges: Two ***non-parallel edges*** are said to be ***adjacent edges*** if they are incident on a common vertex *i.e*, if they have a vertex in common.

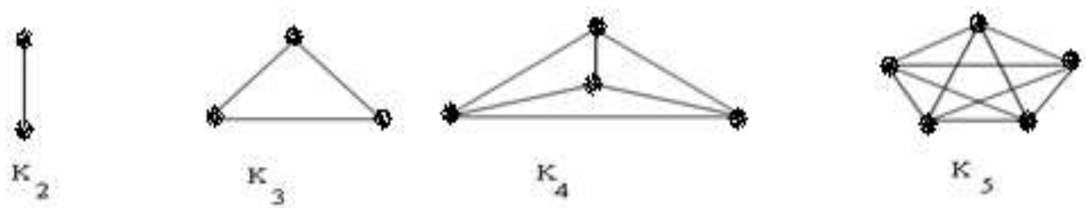
Ex:



In the above graph, A and B are adjacent vertices and e_1, e_2 are adjacent edges.

Complete graph: A simple graph of order >= 2 in which there is an edge between every pair of vertices is called a ***complete graph*** or ***a full graph.*** It is denoted by K_n .

- Complete graph with two, three, four and five vertices are shown in figures below:



- A Complete graph with five vertices is known as Kurtowiski's first graph.

Bipartite graph: Suppose a simple graph G is such that its vertex set V is the union of its mutually disjoint nonempty subsets V_1 and V_2 which are such that each edge in G joins a vertex in V_1 and a vertex in V_2. Then G is called a ***bipartite graph***.

- If E is the edge set of this graph, the graph is denoted by $G = (V_1, V_2, E)$ or $G = G(V_1, V_2, E)$. The sets V_1 and V_2 are called ***bipartites*** or ***partitions*** of the vertex set V.

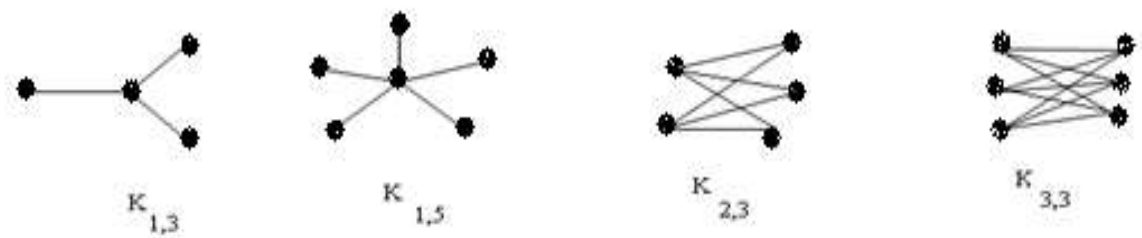
For example, consider the graph G shown below for which the vertex set V = {A, B, C, P, Q, R, S} and the edge set E = {AP, AQ, AR, BR, CS, CQ}.

- Note that V is the union of the subsets V_1 = {A, B, C}, V_2 = {P, Q, R, S} which are such that V_1, V_2 are disjoint, every edge in G joins a vertex in V_1 to a vertex in V_2.

- G contains no edge that joins two vertices both of which are in V_1 or V_2. This graph is a bipartite graph with V_1 = {A, B, C}, V_2 = {P, Q, R, S} as bipartites.

Complete Bipartite graph: A bipartite graph $G = G(V_1, V_2, E)$ is called a ***complete bipartite graph*** if there is an edge between every vertex in V_1 and every vertex in V_2 .

The following figures show some complete bipartite graphs:



- Complete bipartite graph $G = G(V_1, V_2, E)$ in which the bipartites V_1, V_2 contains *r, s* vertices respectively with $r <= s$ is denoted by $K_{r,\, s}$.

- In this graph, each of *r* vertices in V_1 is joined to each of *s* vertices in V_2. Thus $K_{r,\, s}$ contains ***r+s* vertices** and ***rs* edges** i.e., the **order** is ***r+s*** and **size** is ***rs*. Therefore $K_{r,\, s}$ is a (*r*+*s*, *rs*) graph.**

- The graph $K_{3,\, 3}$ is known as Kuratowski's second graph.

Problems:

Q1: If G = G(V, E) is a simple graph, prove that $2\,|E| \le |V|^2 - |V|$

Solution: In a simple graph, there are no multiple edges.

Each edge of a graph is determined by a pair of vertices, *i.e.*, for a pair of vertices, we can have only one edge (2 vertices $\leftrightarrow$ 1 edge).

Hence for a simple graph with *n* >= 2, number of edges cannot exceed number of pair of vertices.

$$i.e.,\ m \le {}^nC_2 \qquad (\because \text{ number of pair of vertices that can be chosen from } n \text{ vertices is } {}^nC_2)$$

$$\Rightarrow m \le \frac{n!}{(n-2)!\,2!}$$

$$\Rightarrow m \le \frac{n(n-1)(n-2)!}{(n-2)!\,2} \qquad \Rightarrow \quad m \le \frac{n(n-1)}{2}$$

$$\Rightarrow 2m \le n^2 - n$$

$$\therefore\ 2\,|E| \le |V|^2 - |V|$$

Q2: Show that a complete graph with *n* vertices, namely K_n has

$\frac{1}{2}n(n-1)$ edges.

Solution: In a complete graph, there exists exactly one edge between every pair of vertices.

Therefore,

the number of edges in a complete graph = the number of pair of vertices.

i.e., $m = {}^nC_2$ ($\because$ number of pair of vertices that can be chosen from n vertices is ${}^nC_2$)

$$\Rightarrow \quad m = \frac{n!}{(n-2)!\,2!}$$

$$\Rightarrow \quad m = \frac{n(n-1)(n-2)!}{(n-2)!\,2}$$

$$\Rightarrow \quad m = \frac{n(n-1)}{2}$$

Thus, in a complete graph with *n* vertices, no. of edges = $m = \frac{1}{2}n(n-1)$.

Q3: Show that a simple graph of order 4 and size 7 and a complete graph of order 4 and size 5 do not exist.

Solution:

(i) By data, order *n* = 4, size = *m* = 7

For a simple graph, $2m \leq n^2 - n$

$$\Rightarrow 2 \times 7 \leq 4^2 - 4$$

$$\Rightarrow 14 \leq 12 \quad (\text{which is } \mathit{false})$$

Thus, a simple graph of order 4 and size 7 does not exist.

(ii) By data, order *n* = 4, size = *m* = 5

For a complete graph, $m = \frac{1}{2}n(n-1)$

$$\Rightarrow 5 = \frac{1}{2}4(4-1)$$

$$\Rightarrow 5 = 6 \quad (\text{which is } \mathit{false})$$

Thus, a complete graph of order 4 and size 5 does not exist.

Q4: (i) How many vertices and how many edges are there in the complete bipartite graphs $K_{4,\,7}$ and $K_{7,\,11}$?

(ii) If the graph $K_{r,\,12}$ has 72 edges, then what is r ?

Solution:

A complete bipartite graph $K_{r,\,s}$ has $r+s$ vertices and rs edges.

(i) The graph $K_{4,\,7}$ has 4+7 = **11 vertices** and 4*7 = **28 edges** and the graph $K_{7,\,11}$ has 7+11 = **18 vertices** and 7*11 = **77 edges**.

(ii) Given that the graph $K_{r,\,12}$ has 72 edges.

Consider $m = rs$

$$72 = r*12$$

This gives $\boldsymbol{r = 6}$.

Q5: Show that a simple graph of order *n* = 4 and size *m* = 5 cannot be a bipartite graph.

Solution:

For a bipartite graph, we have $4m \le n^2 \quad ----(1)$

where order $= |V| = n$ and size $= |E| = m$.

Given $|V| = n = 4$ and $|E| = m = 5$

Substituting in equation (1), we get

$$4 \times 5 \le 4^2$$

i.e., $20 \le 16$ which is false.

Thus a simple graph of order *n* = 4 and size *m* = 5 cannot be a bipartite graph.

REMEMBER:

- For a simple graph, $2m \leq n^2 - n$
- For a complete graph, $m = \frac{1}{2}n(n-1)$
- For a bipartite graph, $4m \leq n^2$
- For a complete bipartite graph $K_{r,s}$, there are $r + s$ vertices and $r*s$ edges.

MATHEMATICAL FOUNDATIONS FOR COMPUTING, PROBABILITY & STATISTICS

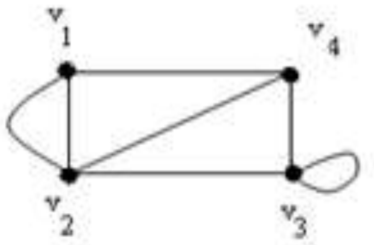
21MATCS41

Module – II : Introduction to Graph Theory

Vertex Degree & Hand Shaking Property

Vertex Degree: Let G = G (V, E) be a graph and *v* be a vertex of G. Then the number of edges of G that are incident on *v* with the loops counted twice is called the ***degree of the vertex v*** and is denoted by ***deg (v) or d(v)***.

- The degrees of the vertices of a graph arranged in ***non – descending*** order is called the ***degree sequence*** of the graph.
- The ***minimum of the degrees*** of vertices of a graph is called the ***degree of the graph.***

Ex: 

$d(v_1) = 3$, $d(v_2) = 4$, $d(v_3) = 4$, $d(v_4) = 3$

Therefore the degree sequence of the graph is 3, 3, 4, 4 and the degree of the graph is 3.

Isolated Vertex, Pendant Vertex & Pendant Edge:

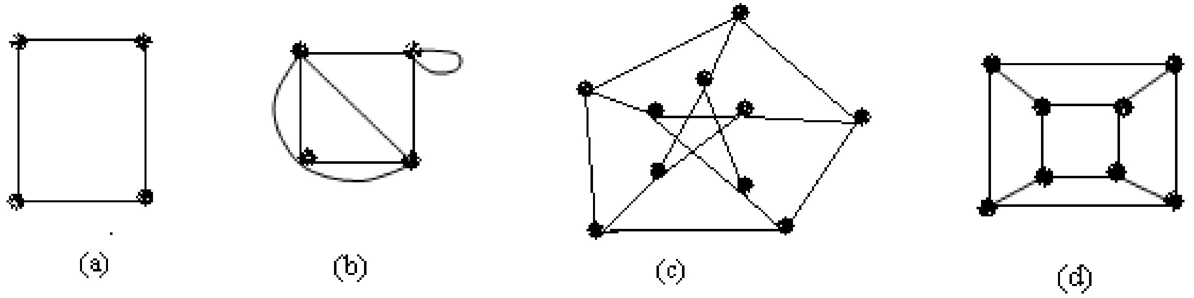
A vertex is an ***isolated vertex*** if and only if its degree is zero.

A vertex of degree 1 is called a ***pendant vertex***.

An edge incident on a pendent vertex is called a ***pendant edge***.

Regular Graph: A graph in which all the vertices are of the same degree k is called a ***regular graph of degree k*** or ***k – regular graph.***

Ex:



The graphs shown in figure (a), (b) are 2 – regular, 4 – regular graphs respectively. A 3 – regular graph is called a ***cubic graph***. The graph shown in figure (c) is a 3 – regular (cubic) graph. This particular cubic graph has 10 vertices and 15 edges, which is called the PETERSEN graph. The graph shown in fig (d) is a cubic graph with $8 = 2^3$ vertices. This particular graph is called 3D hypercube and is denoted by Q_3.

In general, for any positive integer k, a loop free k – regular graph with 2^k vertices is called the ***k – dimensional hypercube (or k – cube)*** and is denoted by Q_k.

Hand Shaking Property: The sum of the degrees of all vertices in a graph is an **even number** and this number is equal to **twice the number of edges** in the graph. *i.e*, for a graph G = (V, E), $\sum_{v\in V}\deg(v) = 2|E|$

Proof: The above property is obvious from the fact that while counting the degrees of vertices, each edge is counted twice(once at each end).

- This property is called the hand shaking property because it states that if several people shake hands, then the total number of handshakes must be even.

Theorem: ***In every graph, the number of vertices of odd degree is even.***

Proof:

Consider a graph with 'n' vertices.

Suppose 'k' of these vertices are of odd degree, then the remaining $n - k$ vertices are of even degree.

Let us assume the 'n' vertices of the form: $v_1, v_2, v_3 \ldots. v_k, v_{k+1}, v_{k+2} \ldots.. v_n$.

Let $v_1, v_2, v_3 \ldots. v_k$ be the vertices of **odd degree** **----- (*)**

and $v_{k+1}, v_{k+2} \ldots. v_n$ be the vertices of **even degree**.

Then $$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) \quad ----- (1)$$

In view of Hand Shaking Property(HSP), the sum on the LHS of equation (1) is equal to twice the number of edges in the graph. And this sum is even.

The second term(sum) in the RHS of equation (1), is the sum of the degree of vertices with even degree each. So, this sum is also even.

Thus the first term(sum) in the RHS of equation (1) must also be even.

***i.e.*,** $\deg(v_1) + \deg(v_2) + \deg(v_3) + \ldots.. + \deg(v_k) = \text{EVEN} \quad ---- (2)$

But from the assumption made(*), each of $\deg(v_1)$, $\deg(v_2)$,.....$\deg(v_k)$ is odd.

Therefore, number of terms in the LHS of equation (2) must be even.

***i.e.*, *k* is even.**

(Since odd numbers added even number of times, the result is even)

Hence the proof.

Problems:

Q1: Can there be a graph with 12 vertices such that two of the vertices have degree 3 each and the remaining 10 vertices have degree 4 each? If so, find its size.

Solution:

Sum of the degree of the vertices $= (2 \times 3) + (10 \times 4)$

$= 46$, which is even.

Hence there can be a graph of the desired type.

To find the size of such graph,

By HSP, Sum of the degree of the vertices = twice the number of edges

$$i.e., \quad 46 = 2|E|$$

$$\Rightarrow |E| = 23$$

Therefore , the size of such graph is 23.

Q2: In a graph G = (V, E), what is the largest possible value for $|\mathrm{V}|$

if $|\mathrm{E}| = 19$ and $\deg(v) \geq 4$ for all $v \in \mathrm{V}$?

Solution:

Given: All the vertices are of degree > = 4

Therefore, Sum of the degrees of vertices > = 4*n*

$$\Rightarrow \quad 2|E| \geq 4n$$

(since by HSP, Sum of the degree of the vertices = Twice the number of edges)

$$\Rightarrow \quad 2 \times 19 \geq 4n$$

$$\Rightarrow \quad 38 \geq 4n$$

$$\Rightarrow \quad n \leq \frac{38}{4} = 9.5 < 10$$

$$i.e., \quad n = |\mathrm{V}| < 10$$

Thus the largest possible value of *n* = 9.

***i.e.*, the given graph can have at most 9 vertices.**

Q3: Show that there is no graph with 12 vertices and 28 edges in the following cases:

(i) The degree of a vertex is either 3 or 4.

(ii) The degree of a vertex is either 3 or 6.

Solution: Suppose there is a graph with 28 edges and 12 vertices, of which k vertices are of degree 3 (each).

(i) If all the remaining $(12 - k)$ vertices have degree 4, then by HSP,

$$3k + 4(12 - k) = 2 \times 28$$

$$\Rightarrow 3k + 48 - 4k = 56$$

$$\Rightarrow -k = 8 \qquad \Rightarrow k = -8 \text{ (\# which is not possible)}$$

(ii) If all the remaining $(12 - k)$ vertices have degree 6, then by HSP,

$$3k + 6(12 - k) = 2 \times 28$$

$$\Rightarrow 3k + 72 - 6k = 56$$

$$\Rightarrow -3k = -16 \qquad \Rightarrow k = \frac{16}{3} \text{ (\# which is not possible)}$$

Thus, the graphs of the desired types cannot exist.

Q4: Show that there exists no simple graphs corresponding to the following degree sequences:

(i) 0, 2, 2, 3, 4 (ii) 1, 1, 2, 3 (iii) 2, 3, 3, 4, 5, 6 (iv) 2, 2, 4, 6

Solution: (i) By HSP, $\sum_{v\in V}\deg(v) = \text{Even number}$

$$\Rightarrow\ 0+2+2+3+4 = 11 \neq \text{Even number}$$

Thus no simple graph of degree sequence 0, 2, 2, 3, 4 exists.

(ii) By HSP, $1+1+2+3 = 7 \neq \text{Even number}$

Thus no simple graph of degree sequence 1, 1, 2, 3 exists.

(iii) By HSP, $2+3+3+4+5+6 = 23 \neq \text{Even number}$

Thus no simple graph of degree sequence 2, 3, 3, 4, 5, 6 exists.

(iii) By HSP, $2+2+4+6 = 14 = \text{Even number}$

But such a graph do not exist because with 4 vertices, we cannot draw a simple graph having degrees 4 and 6.

Q5: (i) If a graph with '*n*' vertices and '*m*' edges is *k* – regular, show that $m=\frac{kn}{2}$

(ii) Does there exist a cubic graph with 15 vertices?

(iii) Does there exist a 4-regular graph with 15 edges?

Solution:

(i) Given that the graph G is *k* – regular.

$\Longrightarrow$ the degree of every vertex is *k*.

Therefore, If G has *n* vertices, then the sum of the degree of vertices is *nk*.

By HSP, this must be equal to 2*m* (if G has *m* edges)

$$i.e.,\ \ nk = 2m \qquad \Rightarrow m=\frac{kn}{2}$$

(ii) If there is a cubic graph(3-regular graph) with 15 vertices, then the number of edges it should have is $m=\frac{kn}{2}$

$$\Rightarrow m = \frac{3\times 15}{2} = \frac{45}{2} \qquad (\#\ \text{since it is not an integer})$$

Thus a cubic graph with 15 vertices does not exist.

(iii) If there is a 4 – regular graph with 15 edges (i.e., $k = 4$, $m = 15$), then the number of vertices it should have is

$$n = \frac{2 \times m}{k} \qquad \left(\sin ce \ m = \frac{kn}{2} \right)$$

$$\Rightarrow n = \frac{2 \times 15}{4} = \frac{30}{4}$$ (# since it is not an integer)

Thus a 4 - regular graph with 15 edges does not exist.

Q6: Determine the order of the graph G = (V, E) in the following cases:
(i) G is a cubic graph with 9 edges.
(ii) G is regular with 15 edges.
(iii) G has 10 edges with 2 vertices of degree 4 and all others of degree 3.

Solution:

(i) Suppose the order of G is '*n*'.

Since G is a cubic graph, all vertices of G have degree 3.

Therefore, Sum of the degrees of the vertices = 3*n*.

By HSP, $\sum_{v\in V} \deg(v) = 2|E|$

$$\Rightarrow \quad 3n = 2\times 9 \qquad \text{(Since G has 9 edges)}$$

$$\Rightarrow \quad n = 6$$

(ii) Given G is a regular graph, then all vertices of G must be of same degree, say *k*.

Suppose the order of G is '*n*'.

Then, Sum of the degrees of the vertices = nk .

By HSP, $\sum_{v \in V} \deg(v) = 2|E|$

$$\Rightarrow \quad nk = 2 \times 15 \qquad \text{(Since G has 15 edges)}$$

$$\Rightarrow \quad n = \frac{30}{k}$$

Since 'k' is a positive integer, it follows that 'n' must be a divisor of 30.

i.e., n = 1, 2, 3, 5, 6, 10, 15, 30 (possible orders of G)

(iii) Suppose the order of G is 'n'.

Given that, 2 vertices of G are of degree 4 and the remaining (n - 2) vertices are of degree 3.

Then by HSP, $\sum_{v \in V} \deg(v) = 2|E|$

$$\Rightarrow \quad (2 \times 4) + (n - 2) \times 3 = 2 \times 10 \qquad \text{(Since G has 10 edges)}$$

$$\Rightarrow \quad 3n - 6 = 12$$

$$\Rightarrow \quad 3n = 18 \qquad \Rightarrow \quad n = 6$$

Q7: For a graph with '*n*' vertices and '*m*' edges, show that $\delta \le \frac{2m}{n} \le \Delta$

where δ is the minimum and Δ is the maximum of the degree of the vertices.

Solution: Let $d_1, d_2, d_3, \ldots.. d_n$ be the degrees of $1^{st}, 2^{nd}, \ldots\ldots n^{th}$ vertex respectively, then by HSP

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$\Rightarrow d_1 + d_2 + d_3 + \ldots.. + d_n = 2m \quad ---- (1)$$

Given that δ is the minimum of $d_1, d_2, d_3, \ldots. d_n$

$$i.e., \ \delta \le d_1, \ \delta \le d_2, \ \delta \le d_3, \ldots\ldots \ \delta \le d_n$$

$$\Rightarrow \delta + \delta + \delta + \ldots\ldots + \delta \le d_1 + d_2 + d_3 + \ldots\ldots. + d_n \quad \text{(here } \delta \text{ is added } n \text{ times)}$$

$$\Rightarrow n\delta \le 2m \quad \text{(from equation 1)}$$

$$\Rightarrow \delta \le \frac{2m}{n} \quad ----- (2)$$

It is also given that Δ is the maximum of $d_1, d_2, d_3, \ldots. d_n$

i.e., $\Delta \geq d_1$, $\Delta \geq d_2$, $\Delta \geq d_3, \ldots\ldots$ $\Delta \geq d_n$

$\Rightarrow$ $\Delta + \Delta + \Delta + \ldots\ldots + \Delta \geq d_1 + d_2 + d_3 + \ldots\ldots. + d_n$ (here Δ is added n times)

$\Rightarrow$ $n\Delta \geq 2m$ (from equation 1)

$$\Rightarrow \quad \Delta \geq \frac{2m}{n}$$

$$i.e., \quad \frac{2m}{n} \leq \Delta \quad ------ \ (3)$$

Thus combining equations (2) and (3), we get

$$\delta \leq \frac{2m}{n} \leq \Delta$$

MATHEMATICAL FOUNDATIONS FOR COMPUTING, PROBABILITY & STATISTICS

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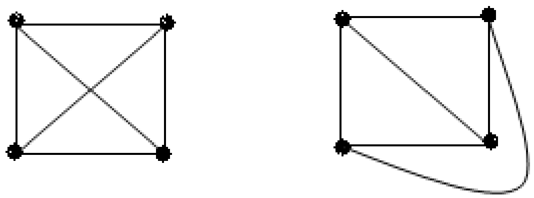
Module – II : Introduction to Graph Theory

Isomorphism

Isomorphism:

Two graphs G and G' are said to be isomorphic to each other if there is a one to one correspondence between their vertices and between their edges such that the adjacency of vertices is preserved. The adjacency of vertices preserved means that for any two vertices u and v in G, if u and v are adjacent in G then the corresponding vertices u', v' in G' are also adjacent in G', then we write $G \cong G'$.



From the definition of the isomorphism of the graphs, it follows that if two graphs are isomorphic, then they must have

(1) The same number of vertices

(2) The same number of edges

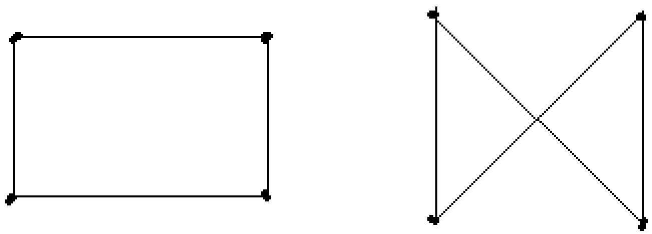
(3) An equal number of vertices with a given degree.

These conditions are necessary but not sufficient.

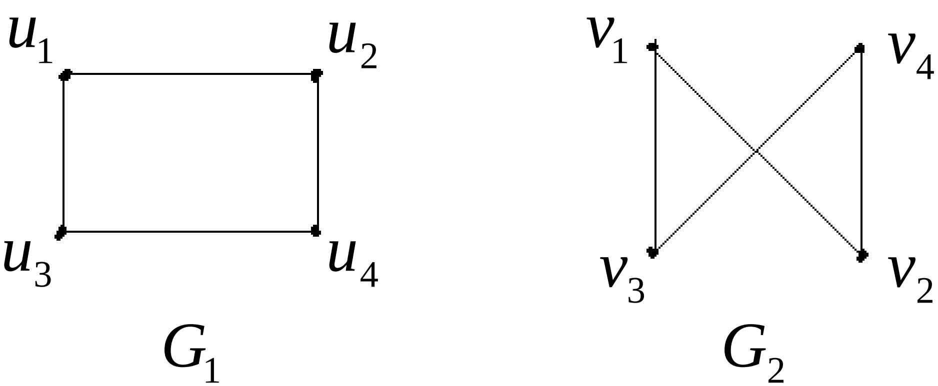
This means that two graphs for which these conditions hold need not be isomorphic. In particular two graphs of the same order and same degree need not be isomorphic.

Problems:

Q1: Verify that the 2 graphs given below are isomorphic:



Solution: Let us rewrite the given graphs and name them as G_1 and G_2. Also we shall name the vertices of the 2 graphs.



The 2 graphs G_1 and G_2 has 4 vertices and 4 edges each.
And the degree of all the vertices in G_1 and G_2 are equal to 2.

We observe that

$$u_1 \leftrightarrow v_1 \qquad u_2 \leftrightarrow v_2 \qquad u_3 \leftrightarrow v_3 \qquad u_4 \leftrightarrow v_4$$

This implies that, there is a 1 – 1 correspondence between the vertices of G_1 and G_2.

Also

$$\{u_1, u_2\} \leftrightarrow \{v_1, v_2\} \qquad \{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$$

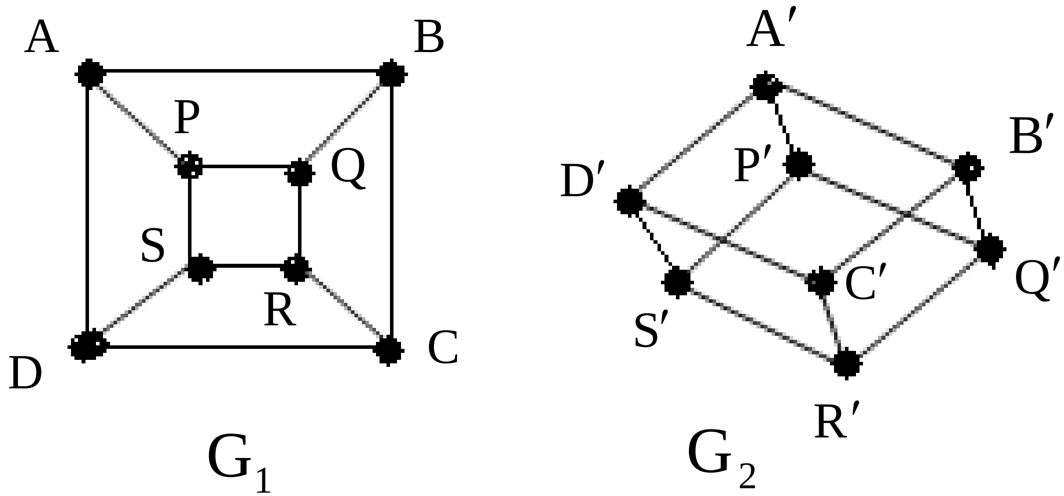
$$\{u_2, u_4\} \leftrightarrow \{v_2, v_4\} \qquad \{u_3, u_4\} \leftrightarrow \{v_3, v_4\}$$

This implies that, there is a 1 – 1 correspondence between the edges of G_1 and G_2.

Also adjacency of the vertices is preserved.

Thus $G_1 \cong G_2$

Q2: Verify the 2 graphs given below are isomorphic:



Solution: The 2 graphs G_1 and G_2 has 8 vertices and 12 edges each.

And the degree of all the vertices in G_1 and G_2 are equal to 3.

We observe that $A \leftrightarrow A'$, $B \leftrightarrow B'$, ... $R \leftrightarrow R'$, $S \leftrightarrow S'$.

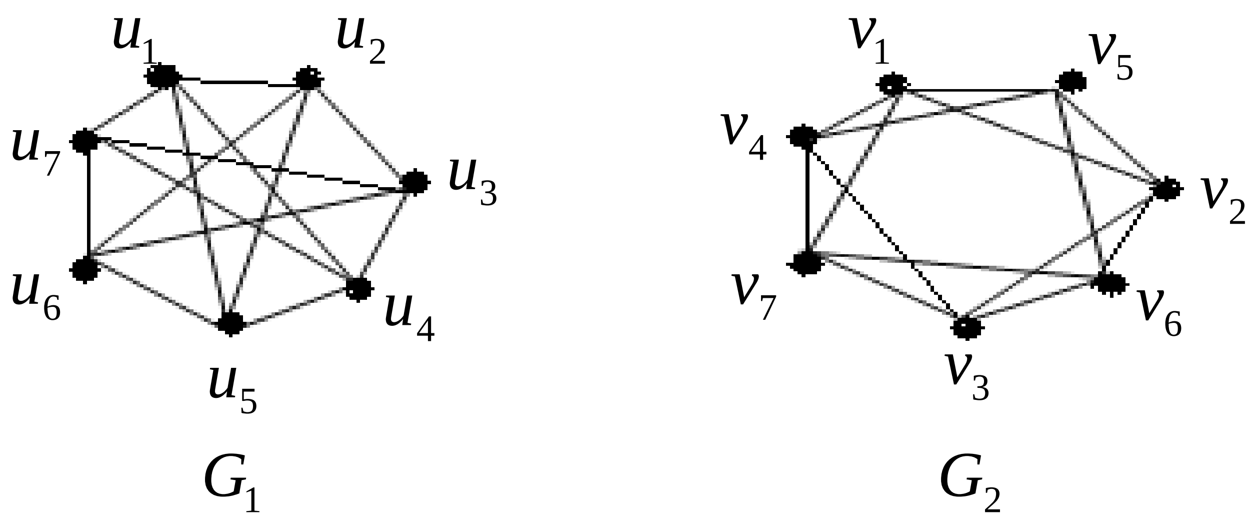
This implies that, there is a 1 – 1 correspondence between the vertices of G_1 and G_2.

Also $\{A, B\} \leftrightarrow \{A', B'\}, \{B, C\} \leftrightarrow \{B', C'\}, ... \{R, S\} \leftrightarrow \{R', S'\}$.

This implies that, there is a 1 – 1 correspondence between the edges of G_1 and G_2.

Also adjacency of the vertices is preserved. **Thus** $G_1 \cong G_2$

Q3: By labeling the graphs show that following graphs are isomorphic:



Solution: The 2 graphs G_1 and G_2 has 7 vertices and 14 edges each.

And the degree of all the vertices in G_1 and G_2 are equal to 4 each.

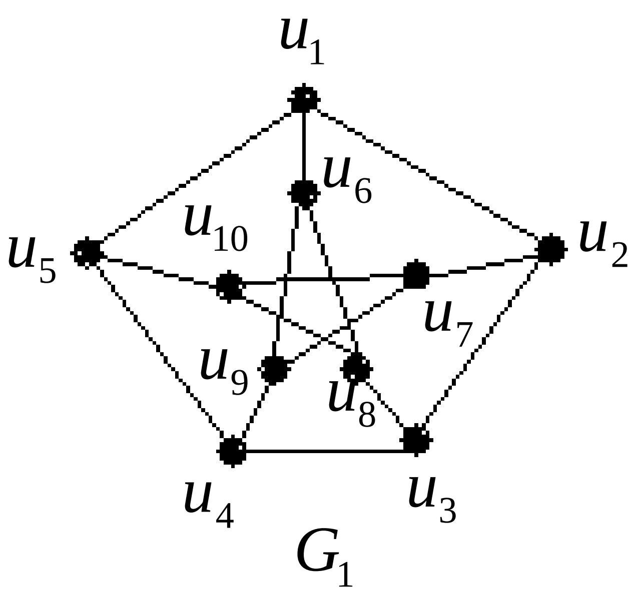
We observe that $u_i \leftrightarrow v_i \quad \forall \ i = 1, 2 \ldots.7$

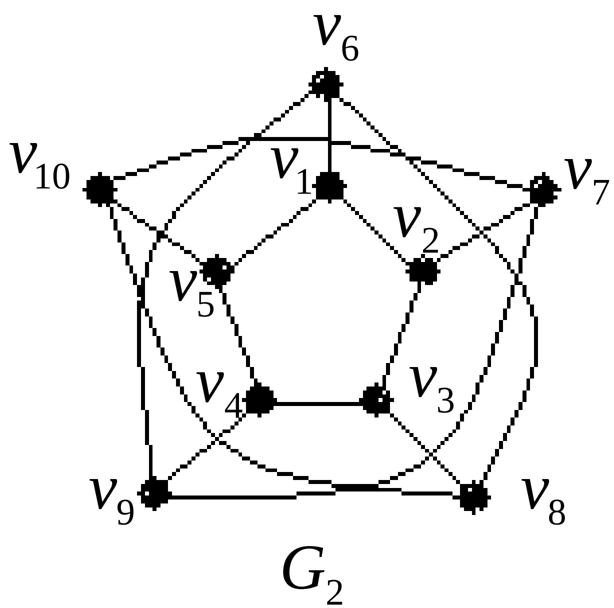
$$\{u_i, u_j\} \leftrightarrow \{v_i, v_j\} \quad \forall \ i = 1, 2, \ldots 7 \text{ and } j = 1, 2, \ldots.7$$

This implies that, there is a 1 – 1 correspondence between the vertices of G_1 & G_2 and there is a 1 – 1 correspondence between the edges of G_1 & G_2. Also adjacency of the vertices is preserved.

$$\therefore \quad G_1 \cong G_2$$

Q4: By labeling the graphs show that following graphs are isomorphic:





Solution: The 2 graphs G_1 and G_2 has 10 vertices and 15 edges each.

And the degree of all the vertices in G_1 and G_2 are equal to 3 each.

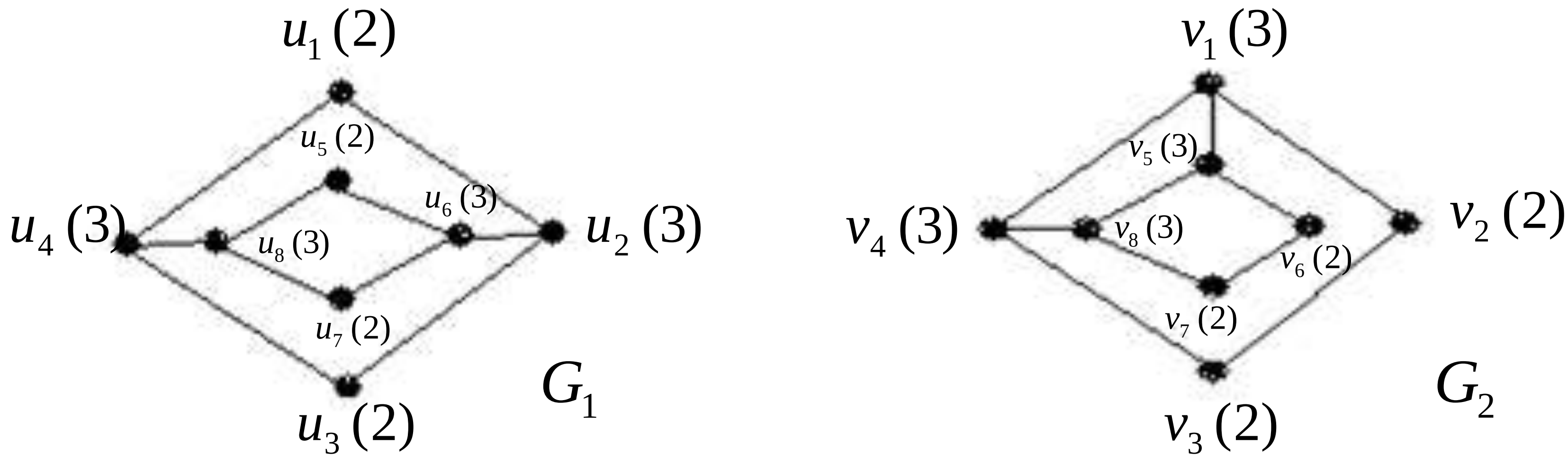
We observe that $u_i \leftrightarrow v_i \quad \forall \ i = 1, 2 \ldots\ldots 10$

$$\{u_i, u_j\} \leftrightarrow \{v_i, v_j\} \quad \forall \ i = 1, 2, \ldots\ldots 10 \text{ and } j = 1, 2, \ldots\ldots 10$$

This implies that, there is a 1 – 1 correspondence between the vertices of G_1 & G_2 and there is a 1 – 1 correspondence between the edges of G_1 & G_2. Also adjacency of the vertices is preserved.

$$\therefore \quad G_1 \cong G_2$$

Q5: Determine whether the following graphs are isomorphic or not.



Solution:

(Let us write the degrees within brackets while naming the vertices)

The 2 graphs G_1 and G_2 has 8 vertices and 10 edges each.

We observe that, in G_1, the vertex u_1 of degree 2 is adjacent to vertices u_2 and u_4 whose degrees are 3 each.

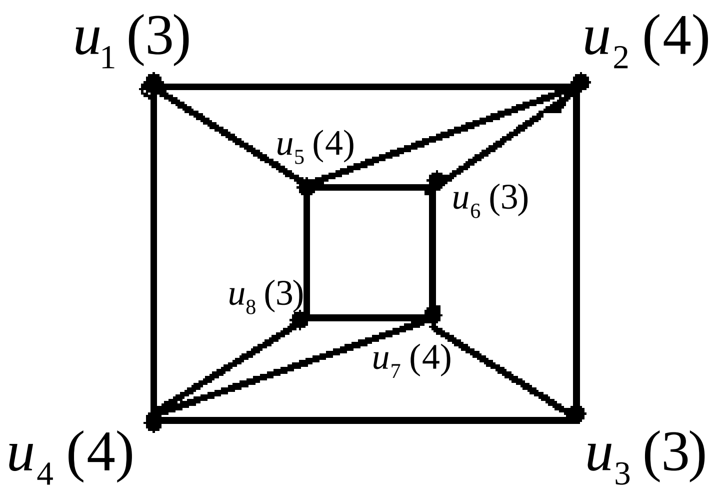
Now suppose, the vertex u_1 in G_1 corresponds to v_2 in G_2(since both are of same degree)

Then we observe that, in G_2 , the vertex v_2 of degree 2 is adjacent to vertex v_1 of degree 3 and vertex v_3 of degree 2.

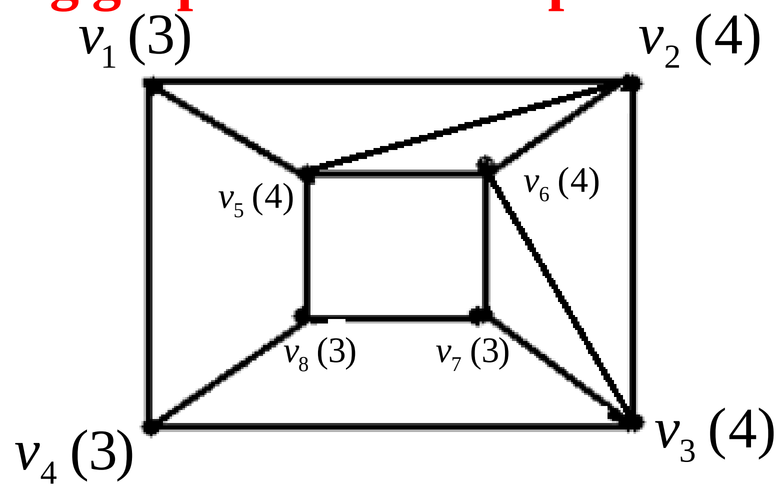
This implies that, the adjacency of vertices is not preserved.

Hence G_1 is NOT ISOMORPHIC to G_2.

Q6: Determine whether the following graphs are isomorphic or not.



G_1



G_2

Solution:

(Let us write the degrees within brackets while naming the vertices)

The 2 graphs G_1 and G_2 has 8 vertices and 14 edges each.

We observe that, the vertex u_1 of degree 3 in the graph G_1 is adjacent to vertices u_2, u_4, u_5, all of which are of degrees 4 each.

Now suppose, the vertex u_1 in G_1 corresponds to v_1 in G_2(since both are of same degree)

Then we observe that, in G_2 , the vertex v_1 is adjacent to v_2, v_4 and v_5 whose degrees are 4, 3, 4 respectively.

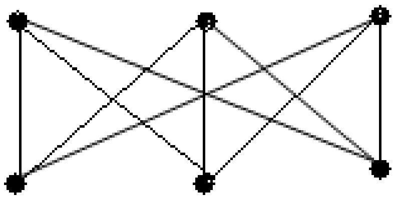
This implies that, the adjacency of vertices is not preserved.

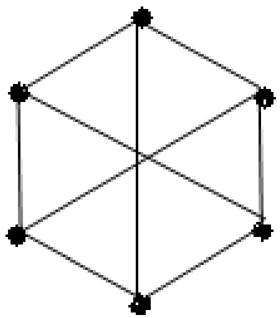
Hence G_1 is NOT ISOMORPHIC to G_2.

PRACTICE QUESTIONS:

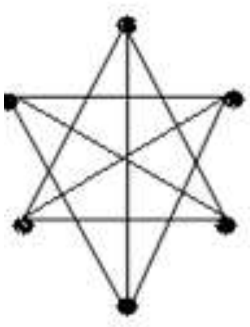
Verify the following graphs are Isomorphic or not?

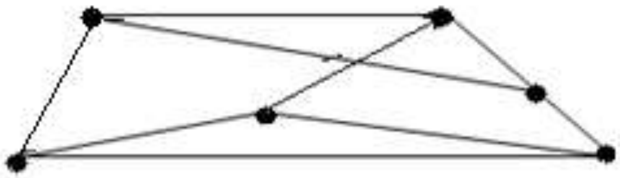
1.



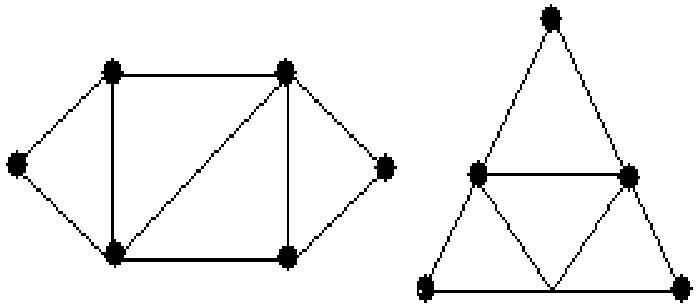


2.

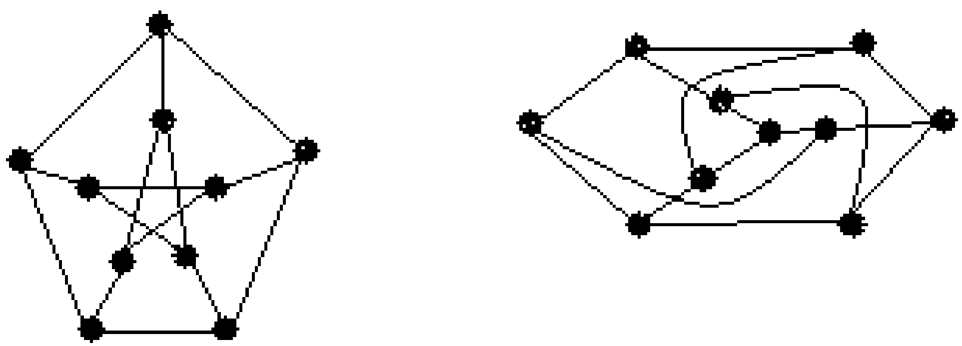




3.



4.



MATHEMATICAL FOUNDATIONS FOR COMPUTING, PROBABILITY & STATISTICS

21MATCS41

Module – II : Introduction to Graph Theory

Sub graphs and Complement of a graph

Sub graphs:

Given two graphs G, G_1 ,we say that G_1 is a ***sub graph*** of G if the following conditions are satisfied:

(i) All the vertices and edges of G_1 are in G.

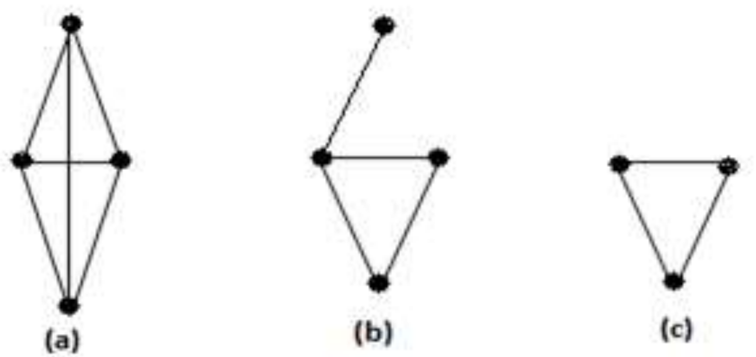
(ii) Each edge of G_1 has the same end vertices in G as in G_1.

Spanning Sub graph:

A sub graph G_1 of a graph G is a spanning sub graph of G whenever the vertex set of G_1 contains all the vertices of G.

Thus, a graph and all its spanning sub graphs have the same vertex set.

Ex:



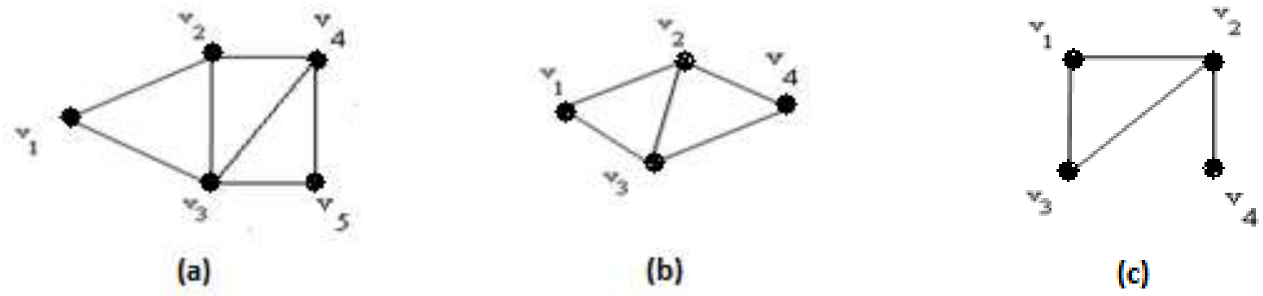
For the graph shown in fig (a), the graph in fig (b) is a spanning sub graph whereas the graph in fig (c) is a sub graph, but not a spanning sub graph.

Induced Sub graph:

Given a graph G = (V, E), if there is a sub graph $G_1 = (V_1, E_1)$ of G such that every edge {A, B} of G is an edge of G_1 also, where $A, B \in V_1$

Then G_1 is called an **induced sub graph** of G (induced by V_1) and is denoted by $\langle V_1 \rangle$

Ex:



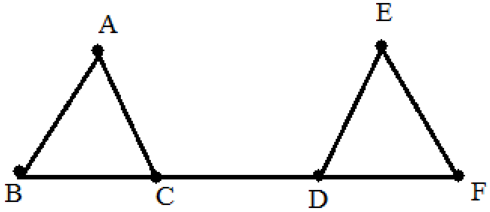
For the graph shown in fig (a), the graph shown in fig (b) is an induced sub graph, induced by the vertex set $V_1 = \{v_1, v_2, v_3, v_4\}$ whereas the graph shown in fig (c) is not an induced sub graph since there is no edge between v_3 and v_4.

Edge disjoint and Vertex disjoint Sub graphs:

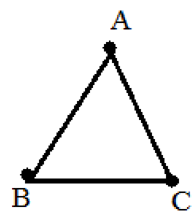
Let G be a graph and G_1 and G_2 be two sub graphs of G then,

(i) G_1 and G_2 are said to be ***edge disjoint*** if they do not have any common edge. (ii) G_1 and G_2 are said to be ***vertex disjoint*** if they do not have any common edge and any common vertex.

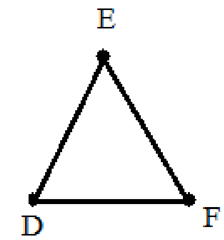
Ex 1:



G



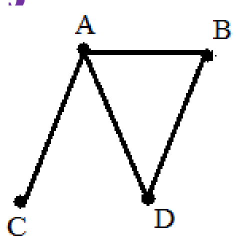
G_1



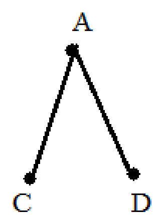
G_2

For the graph G, the graphs G_1 and G_2 are vertex disjoint, as well as edge disjoint sub graphs.

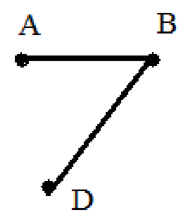
Ex 2:



G



G_1



G_2

The graphs G_1 and G_2 are edge disjoint, but not vertex disjoint since A and D are common vertices in G_1 and G_2.

OPERATIONS ON GRAPHS

Consider two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Then the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$ is called the union of G_1, G_2 and is denoted by $G_1 \cup G_2$.

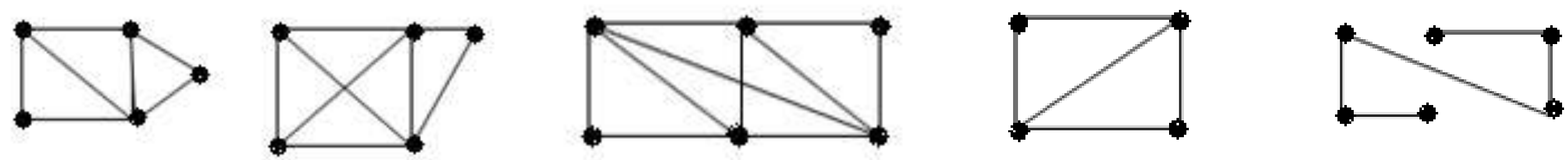
Thus $G_1 \cup G_2 = \left(V_1 \cup V_2, E_1 \cup E_2\right)$

If $V_1 \cap V_2 \neq \phi$ then the graph whose vertex set is $V_1 \cap V_2$ and the edge set is $E_1 \cap E_2$, is called the intersection of G_1, G_2 and is denoted by $G_1 \cap G_2$.

Thus $G_1 \cap G_2 = \left(V_1 \cap V_2, E_1 \cap E_2\right)$ if $V_1 \cap V_2 \neq 0$

The graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \Delta E_2$ where $E_1 \Delta E_2$ is the symmetric difference of E_1 , E_2. (The symmetric difference $E_1 \Delta E_2$ denotes the set of all those elements (here edges) which are in E_1 or E_2 but not in both) This graph is called the ***ring sum*** of G_1, G_2 and it is denoted by $G_1 \Delta G_2$. Thus $G_1 \Delta G_2 = \left(V_1 \Delta V_2, E_1 \Delta E_2\right)$.

For the graphs shown in figures below, their union, intersection and the ring sum is shown:



G_1 G_2 $G_1 \cup G_2$ $G_1 \cap G_2$ $G_1 \Delta G_2$

Decomposition: A graph G is **decomposed** or **partitioned** into two sub graphs G_1, G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi =$ Null graph

Deletion: If v is a vertex in a graph G, then G – v denotes the graph obtained by deleting v and all the edges incident on v, from G.

This graph G – v, is referred as ***vertex deleted sub graph of G.***

Deletion of a vertex always results in the deletion of all the edges incident on that vertex. G – v is a sub graph of G induced by

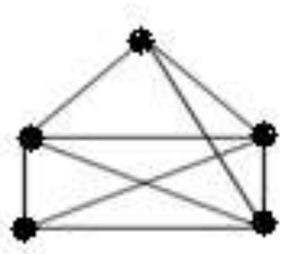
$\mathrm{V}_1 = \mathrm{V} - \{v\}$.

If *e* is an edge in a graph G, then G – *e* denotes the sub graph of obtained by deleting *e* (but not its end vertices) from G.

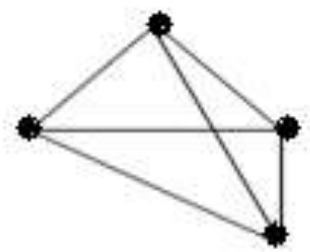
This sub graph is called ***edge – deleted sub graph of G.***

The deletion of an edge does not alter the number of vertices. As such, an edge deleted sub graph of a graph is a spanning sub graph of the graph.

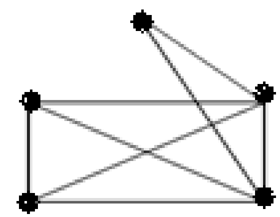
For the graph G shown in fig (a), the sub graphs G – *v* and G – *e* are shown in fig (b) and fig (c) respectively.



(a): G



(b): G - v



(c): G - e

Complement of a Simple graph: Complement of a simple graph G is the graph obtained by deleting those edges which are in G and adding the edges which are not in G, usually denoted by $\overline{G}$.

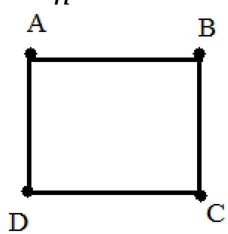
Note: 1) G and $\overline{G}$ have the same vertex set.

2) Two vertices are adjacent in G iff they are not adjacent in $\overline{G}$.

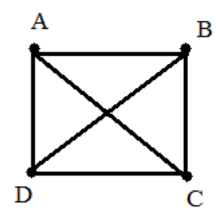
3) Complement of a complete graph K_n is a null graph.

4) $\overline{G} = K_n - G$.

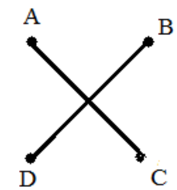
Ex:



G



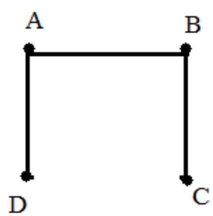
K_4



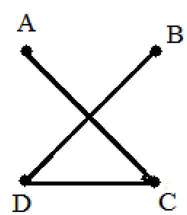
$\overline{G}$

Self Complementary graphs: A simple graph G which is isomorphic to its complement is called a self complementary graph.

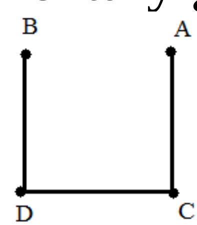
Ex:



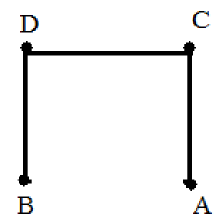
G



$\overline{G}$



$\overline{G}$



$\overline{G}$

$\Rightarrow G \cong \overline{G}$

PROBLEMS:

Q1: Let G be a simple graph of order '*n*'. If the number of edges in G is 56 and in $\overline{G}$ is 80, then what is the value of n?

Solution:

We have $\overline{G} = K_n - G$

$\therefore$ number of edges in $\overline{G}$ = number of edges in K_n − number of edges in G

$$\Rightarrow \quad 80 = \frac{1}{2}n(n-1) - 56$$

$$\Rightarrow \quad (56+80)\times 2 = n^2 - n$$

$$\Rightarrow \quad n^2 - n - 272 = 0$$

Solving, we get $n = 17$, $n = -16$.

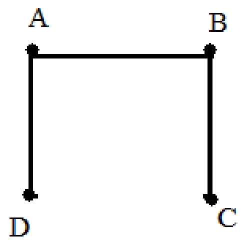
But, n is the number of vertices in the given simple graph G, which cannot be negative.

Thus order of G = *n* =17.

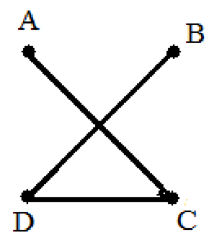
Q2: Find an example of a self complementary graph on 4 vertices and one on 5 vertices.

Solution: We know that a simple graph G which is isomorphic to its complement is called a self complementary graph.

Example on 4 vertices:

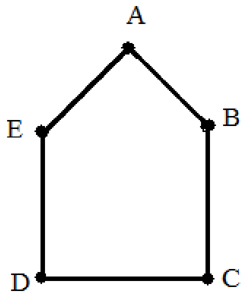


G

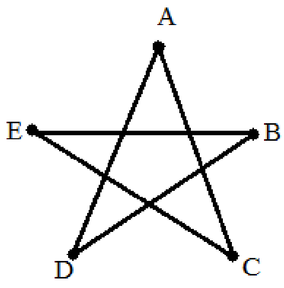


$\overline{G}$

Example on 5 vertices:



G



$\overline{G}$